The Formal Demography of Kinship: Age-Specific and Multistate Models

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A paper for evaluation of a talk submitted to EPC 2020

Hello. This large-ish document is intended to provide extra information to supplement my abstract submitted for presentation at EPC 2020. It is a combination of two papers.

- The first appeared last month in *Demographic Research.* It describes the age-specific kinship model and shows a sampling of the range of results that it can produce. My submitted abstract will be the first presentation of the model in Europe.

- The second document is a draft of a paper that derives the multistate generalization of the age-specific model, with an application to age-parity dynamics of kinship.

My presentation will present aspects (but not details, of course) of both papers.
Enjoy!

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Research Article

The formal demography of kinship: A matrix formulation

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The formal demography of kinship: A matrix formulation

Hal Caswell[1]

Abstract

BACKGROUND
Any individual is surrounded by a network of kin that develops over her lifetime. In a justly famous paper, Goodman, Keyfitz, and Pullum (1974) presented formal calculations of the mean numbers of (female, matrilineal) kin implied by a mortality and fertility schedule.

OBJECTIVE
The aim of this paper is a new theory of kinship demography that provides age distributions as well as expected numbers, permits calculation of properties (e.g., dependency) of kin, is easily computable, and does not require simulation.

METHODS
The analysis relies on a novel application of the matrix formulation of cohort component population projection to describe the dynamics of a kinship network. The approach arises from the observation that the kin of a focal individual form a population, and can be modelled as one.

RESULTS
Kinship dynamics are described by a coupled system of non-autonomous matrix equations. I show how to calculate age distributions, total numbers, prevalence, dependency, and the experience of the death of relatives. As an example, I compare the kinship networks implied by the period vital rates of Japanese women in 1947 and 2014. Over this interval, fertility declined by 70% while life expectancy increased by 60%. The implications of these changes for kinship structure are profound; a lifetime dominated, under 1947 rates, by the experience of the death of kin has changed to one in which the death of kin is a rare event. On the other hand, the burden of dependent aged kin, including those suffering from dementia, is many-fold larger under 2014 rates.

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CONCLUSIONS
This new theory opens to investigation hitherto inaccessible aspects of kinship, with potential applications to many problems in family demography.

1. Introduction

Birth and death are universals of demography. Every individual, without exception, will eventually die. Every individual, without exception, was born and most individuals will have the experience of producing children during their lives. No surprise then, that there exists a rich and powerful formal demographic theory of mortality, fertility, and how their interactions determine population growth and structure.

The third universal of human demography is kinship and family. The children of humans are unusually dependent, compared to other species (Hrdy 2009), and every individual human has some experience of family (or an attempted institutional substitute, as in orphanages). These family interactions reflect, in various ways in different cultures, the degrees of kinship among individuals. The development of a formal demography of kinship and families is challenging, because it requires accounting not only for individuals, but also for relations among individuals.

The analysis of kinship is a venerable problem (e.g., Greenwood and Yule 1914; Lotka 1931).[2] The modern approach to kinship was derived in a justly famous paper by Goodman, Keyfitz, and Pullum (1974; see also Keyfitz and Caswell 2005: Chap. 15). Their analysis takes as input an age schedule of mortality and fertility, and calculates from these schedules the mean numbers of specified kin [daughters, granddaughters (and further generations of descendants), mothers, grandmothers (and more remote generations of ancestors), sisters, nieces, maternal aunts, and cousins] of an individual at a specified age x. Their methodology is a tour de force of multiple integration over the survival and reproduction of all individuals involved in a type of kin, tracking the routes by which individuals of one type can produce surviving individuals of another type. Later extensions have led to more elaborate integral formulations (Krishnamoorthy 1979). Alternative calculations have been presented by Burch (1995), and important stochastic extensions by Pullum (Pullum 1982; Pullum and Wolf 1991).

As powerful as it is, the approach of Goodman, Keyfitz, and Pullum (1974) has limitations. It provides numbers of kin, but not their age distributions. It provides mean numbers of kin, but not variances or covariances. It describes living kin, but provides no information on the dead. It relies on age-classified vital rates, and does not generalize easily to stage-classified or multistate models. Its implementation requires multiple

[2] Perhaps the early interest in kinship was motivated because, in 1914, much of the world was ruled, at least nominally, by hereditary monarchs, a context in which kinship is of central political importance.

integrals to be approximated by high dimensional summations (Goodman, Keyfitz, and Pullum 1974) with a confusing proliferation of subscripts. This paper is the first report on a new approach to kinship demography that overcomes these limitations.

Kinship and kinship structures appear in diverse applications throughout demography (and, although it is not the focus here, population biology; see Tanskanen and Danielsbacka 2019). To cite just a few examples, consider (1) intergenerational transfers by bequests (Zagheni and Wagner 2015; Brennan, James, and Morrill 1982); (2) economic support for kin, including support of grandparents by children and grandchildren (e.g., Stecklov 2002; Wachter 1997; Tu, Freedman, and Wolf 1993; Himes 1992) and grandparents acting as a safety net for grandchildren (Bengtson 2001); (3) intergenerational reproductive conflict as a factor in the evolution of menopause (Lahdenperä et al. 2012; Croft et al. 2017); (4) the estimation of demographic parameters from limited data (Harpending and Draper 1990; McDaniel and Hammel 1984; Goldman 1978); (5) the medical and psychological implications of the experience of death of close kin (Umberson et al. 2017); (6) changes in generational overlap as populations age (Dykstra 2010); (7) social unrest fueled by the age distribution of children within families in societies where children of different orders have different social roles (Roche 2010, 2014); (8) "sandwich" families, where individuals care for both dependent children and aging parents (DeRigne and Ferrante 2012); (9) "boomerang" families in which adult children return to live with parents (Farris 2016); (10) orphanhood (e.g., due to HIV/AIDS) and its attendant social consequences (Jones and Morris 2003; Zagheni 2010; Kazeem and Jensen 2017); (11) the interaction of population aging and the likelihood of living ancestors (Gisser and Ediev 2019); and (12) intergenerational social mobility (Song 2016; Song and Mare 2017; Song and Campbell 2017; Mare and Song 2015).

This paper presents a new formulation of the demography of kinship. It provides not only the mean numbers of kin of an individual of any age, but also age distribution of the kin and a variety of demographic properties calculated from those distributions. It also calculates the experience of the death of kin and their ages at death.

Notation: In what follows, matrices are denoted by upper case bold characters (e.g., $\mathbf{U}$) and vectors by lower case bold characters (e.g., $\mathbf{a}$). Vectors are column vectors by default; $\mathbf{x}^{\mathsf{T}}$ is the transpose of $\mathbf{x}$. The ith unit vector (a vector with a 1 in the ith location and zeros elsewhere) is $\mathbf{e}_i$. The vector $\mathbf{1}$ is a vector of ones, and $\mathbf{I}$ is the identity matrix. The symbol $\circ$ denotes the Hadamard, or element-by-element product (implemented by .* in MATLAB and by * in R). The notation $\|\mathbf{x}\|$ denotes the 1-norm of $\mathbf{x}$. When necessary, subscripts may be used to denote the size of a vector or matrix; e.g., $\mathbf{I}_\omega$ is an identity matrix of size $\omega \times \omega$. On occasion, MATLAB notation will be used to refer to rows and columns; e.g., $\mathbf{F}(i,:)$ and $\mathbf{F}(:,j)$ referring to the ith row and jth column of the matrix $\mathbf{F}$.

2. The demography of kinship

Introducing Focal. The analysis is organized in terms of the kin of a focal individual. This individual appears so often as to deserve a name, so I will refer to her/him as Focal. Focal is an individual of a specified age and sex (female, for this paper), who might also be characterized by other properties, such as education, health, partnership status, parity, etc. Focal is a member of a population subject to a mortality and fertility schedule, and by any age will have developed a network of kin of different kinds and degrees of relatedness. The kin are the product of the reproduction of Focal (in the case of children), or of other kin (e.g., the sisters of Focal are the children of Focal's mother). In this paper, as in Goodman, Keyfitz, and Pullum (1974), calculations refer to female kin through female lineages.

The analysis here, like that of Goodman, Keyfitz, and Pullum (1974), makes three assumptions: (1) Homogeneity. All individuals in the population are subject to the same schedules of mortality and fertility. (2) Time invariance. The vital rates to which the individuals are subject do not change, and have not changed, over time. (3) Stability. The population is at the stable age (or age$\times$stage) structure implied by the mortality and fertility schedules. This assumption is implied by the assumptions of homogeneity and time invariance.

To relax the time-invariance assumption would require writing quantities as joint functions of time and the age of Focal, and will not be considered here. To relax the homogeneity assumption would require enlarging the i-state space to include the numbers and ages of kin of different kinds, each with its own rates. This will be pursued elsewhere. The stability assumption is used to obtain the mixing distribution of the ages of the mothers of Focal at the time of her birth. This could be relaxed by using an empirically measured distribution of ages of mothers.

The population of which Focal is a part is characterized by a mortality and a fertility schedule. The mortality schedule is incorporated into a matrix $\mathbf{U}$, of dimension $\omega \times \omega$, with survival probabilities on the subdiagonal and zeros elsewhere. The fertility schedule is incorporated into a matrix $\mathbf{F}$, of dimension $\omega \times \omega$, with effective fertility on the first row and zeros elsewhere. For example, if $\omega = 3$,

$$\mathbf{U} = \left(\begin{array}{ccc} 0 & 0 & 0 \\ p_1 & 0 & 0 \\ 0 & p_2 & [p_3] \end{array} \right) \qquad \mathbf{F} = \left(\begin{array}{ccc} f_1 & f_2 & f_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right). \tag{1}$$

The optional entry in the ω, ω position in $\mathbf{U}$ describes an open final age interval. Effective fertility refers to the production of daughters. Stage-classified models would lead to other structures for $\mathbf{U}$ and $\mathbf{F}$. The population projection matrix describing Focal's population is

$$\mathbf{A} = \mathbf{U} + \mathbf{F}. \tag{2}$$

It has the familiar Leslie matrix structure, with non-zero entries only on the subdiagonal and the first row (e.g., Leslie 1945; Caswell 2001).

The vital rates in $\mathbf{A}$ imply an asymptotic population growth rate λ given by the dominant eigenvalue of $\mathbf{A}$ (or the corresponding continuous-time rate $r = \log \lambda$), and a stable age distribution given by the associated right eigenvector $\mathbf{w}$, scaled to sum to 1. The net reproductive rate R_0 is given by the dominant eigenvalue of the matrix $\mathbf{F}\left(\mathbf{I} - \mathbf{U}\right)^{-1}$.

An important role in kinship calculations is played by the distribution of the ages of the mothers of offspring produced in the population, which is denoted $\boldsymbol{\pi}$. Here, this distribution is taken to be that implied by the stable population, which is given by

$$\boldsymbol{\pi} = \frac{\mathbf{F}(1,:)^{\mathsf{T}} \circ \mathbf{w}}{\|\mathbf{F}(1,:)^{\mathsf{T}} \circ \mathbf{w}\|} \tag{3}$$

The mean age over this distribution is the generation time (Coale 1972). Other distributions could be substituted for this stable population if desired.

2.1 The kin of Focal are a population

The key to the what follows is the recognition that *the kin, of any specified degree, of Focal comprise a population*, albeit one with some special properties. Being a population, the kin might as well be modeled as such. This deceptively simple observation is key to the analysis.

Let the vector $\mathbf{k}(x)$ denote the age distribution of the population of some specified type of kin, at age x of Focal. This vector $\mathbf{k}(x)$ contains the survivors of the population at Focal's age $x - 1$, with survival accounted for by the matrix $\mathbf{U}$.

The kin of Focal subsidized population. That is, new members of the population arise not from reproduction of current members, but from elsewhere (Pascual and Caswell 1991; Caswell 2008).[3] For example, new daughters of Focal do not arise from reproduction of current daughters (those are grand-daughters), but from the reproduction of Focal.

The kin of Focal at birth provide the initial condition for the dynamics. This initial condition, $\mathbf{k}(0) = \mathbf{k}_0$, depends on the type of kin considered. Focal will, for example, have no daughters at birth, but may very well have older sisters.

Combining survival, subsidy, and initial conditions yields the model for the dynamics of the kin $\mathbf{k}(x)$:

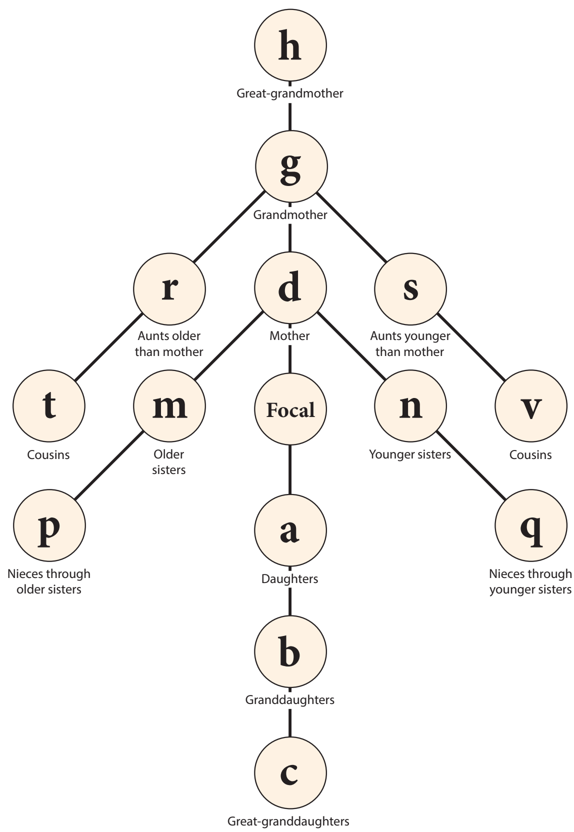
$$\mathbf{k}(x+1) = \mathbf{U}\mathbf{k}(x) + \boldsymbol{\beta}(x) \tag{4}$$

$$\mathbf{k}(0) = \mathbf{k}_0 \tag{5}$$

[3] Subsidy is common in species with widely dispersed offspring, such as many marine invertebrates, and also appears in models of recruitment to organizations (e.g., Pollard 1968); such systems are referred to as 'open' by Bartholomew (1982). Now subsidy appears also in the dynamics of kin.

where x is the age of Focal and $\boldsymbol{\beta}(x)$ is a vector giving the age distribution of the subsidy of these kin at age x of Focal.

Figure 1: The kinship network. The network of kin defined in Goodman, Keyfitz, and Pullum (1974) and Keyfitz and Caswell (2005). The symbols (a, b, etc.) are used here to denote the age distribution vectors of each type of kin of Focal. That is, e.g., $\mathrm{a}(x)$ is the expected age distribution of daughters at age x of Focal.



Focal is surrounded by a network of kin of different types and different degrees of relatedness. My goal here is to describe the dynamics of this network; the model is a coupled system of non-autonomous matrix difference equations of the form (4) and (5).

Figure 1, modified from Goodman, Keyfitz, and Pullum (1974), shows a portion of this network. I consider only direct matrilineal descent (mothers, daughters, granddaughters, etc.) and only consanguineal relationships. Each of these 14 types of kin is described by a population vector ($\mathbf{a}(x)$, $\mathbf{b}(x)$, $\ldots$), as indicated in Figure 1. Keeping track of 14 types of kin poses notational challenges, because some symbols need to be used for other purposes. The rationale behind the exclusion of some letters from the assignments in Figure 1 is as follows. The symbol $\mathbf{e}_j$ is already in use as the jth unit vector (i.e., a vector with a 1 in the jth entry and zeros elsewhere), $\mathbf{F}$ is the fertility matrix, i and j are reserved for indices and counters, $\mathbf{k}$ is used to refer to a generic kin, ℓ is the survivorship function, $\mathbf{o}$ is generally confusing as a symbol, $\mathbf{U}$ is the transition and survival matrix, $\mathbf{w}$ the stable age distribution, and x is age.

The network in Figure 1 can be extended further in the direction of descendants, ancestors, and chains derived from the siblings of ancestors (as, for example, cousins are the descendants of the siblings of the mother of Focal). I will discuss some of these descendants below.

Armed with these definitions and the general model in (4) and (5), we can proceed to derive models for the dynamics of each type of kin.

2.1.1 Daughters and descendants

Each type of descendent depends on the reproduction of another type of descendent, or of Focal herself.

$\mathbf{a}(x)$ **= daughters of Focal.** Daughters are the result of the reproduction of Focal. Since Focal is assumed to be alive at age x, the subsidy vector is $\boldsymbol{\beta}(x) = \mathbf{F}\mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector for age x. Because we may be sure that Focal has no daughters when she is born, the initial condition is $\mathbf{a}_0 = \mathbf{0}$. Thus

$$\mathbf{a}(x+1) = \mathbf{U}\mathbf{a}(x) + \mathbf{F}\mathbf{e}_x \tag{6}$$

$$\mathbf{a}_0 = \mathbf{0}. \tag{7}$$

$\mathbf{b}(x)$ **= granddaughters of Focal.** Granddaughters are the children of the daughters of Focal. At age x of Focal, these daughters have age distribution $\mathbf{a}(x)$, so $\boldsymbol{\beta}(x) = \mathbf{F}\mathbf{a}(x)$. Because Focal has no granddaughters at birth, the initial condition is $\mathbf{0}$;

$$\mathbf{b}(x+1) = \mathbf{U}\mathbf{b}(x) + \mathbf{F}\mathbf{a}(x) \tag{8}$$

$$\mathbf{b}_0 = \mathbf{0}. \tag{9}$$

$\mathbf{c}(x)$ **= great-granddaughters of Focal.** Similarly, great-granddaughters are the result of reproduction by the granddaughters of Focal, with an initial condition of $\mathbf{0}$.

$$\mathbf{c}(x+1) = \mathbf{U}\mathbf{c}(x) + \mathbf{F}\mathbf{b}(x) \tag{10}$$

$$\mathbf{c}_0 = \mathbf{0}. \tag{11}$$

The extension to arbitrary levels of direct descendants is obvious. Let $\mathbf{k}_n$, in this case, be the age distribution of descendants of level n, where $n = 1$ denotes children. Then

$$\mathbf{k}_{n+1}(x+1) = \mathbf{U}\mathbf{k}_{n+1}(x) + \mathbf{F}\mathbf{k}_n(x) \tag{12}$$

with the initial condition

$$\mathbf{k}_{n+1}(0) = \mathbf{k}_n(0) = 0$$

2.1.2 Mothers and ancestors

The surviving mothers and other direct ancestors depend on the age of those ancestors at the time of the birth of Focal.

$\mathbf{d}(x)$ **= mothers of Focal.** The population of mothers of focal consists of at most a single individual (step-mothers are not considered here). It has an expected age distribution, and is subject to survival according to $\mathbf{U}$. No new mothers arrive after Focal's birth, so the subsidy term is $\boldsymbol{\beta}(x) = \mathbf{0}$.
At the time of Focal's birth, she has exactly one mother, but we do not know her age. Hence the initial age distribution $\mathbf{d}_0$ of mothers is a mixture of unit vectors $\mathbf{e}_i$; the mixing distribution is the distribution $\boldsymbol{\pi}$ of ages of mothers given by (3). Thus,

$$\mathbf{d}(x+1) = \mathbf{U}\mathbf{d}(x) + \mathbf{0} \tag{13}$$

$$\mathbf{d}_0 = \sum_i \pi_i \mathbf{e}_i = \boldsymbol{\pi}. \tag{14}$$

$\mathbf{g}(x)$ **= grandmothers of Focal.** The grandmothers of Focal are the mothers of the mother of Focal. No new grandmothers appear, so once again the subsidy term $\boldsymbol{\beta}(x) = \mathbf{0}$. The age distribution of grandmothers at the birth of Focal is the age distribution of the mothers of Focal's mother, at the age of Focal's mother when Focal is born. The age of Focal's mother at Focal's birth is unknown, so the initial age distribution of grandmothers is a mixture of the age distributions $\mathbf{d}(x)$ of mothers, with mixing distribution $\boldsymbol{\pi}$:

$$\mathbf{g}(x+1) = \mathbf{U}\mathbf{g}(x) + \mathbf{0} \tag{15}$$

$$\mathbf{g}_0 = \sum_i \pi_i \mathbf{d}(i). \tag{16}$$

$\mathbf{h}(x)$ **= great-grandmothers of Focal.** Again, the subsidy term is $\boldsymbol{\beta}(x) = \mathbf{0}$. The initial condition is a mixture of the age distributions of the grandmothers of Focal, with mixing distribution $\boldsymbol{\pi}$:

$$\begin{aligned} \mathbf{h}(x+1) &= \mathbf{U}\mathbf{h}(x) + \mathbf{0} && (17) \\ \mathbf{h}_0 &= \sum_i \pi_i \mathbf{g}(i). && (18) \end{aligned}$$

The extension to arbitrary levels of direct ancestry is clear. Let $\mathbf{k}_n$ be, in this case, the age distribution of ancestors of level n, where $n = 1$ denotes mothers. Then the dynamics and initial conditions are

$$\begin{aligned} \mathbf{k}_{n+1}(x+1) &= \mathbf{U}\mathbf{k}_{n+1}(x) + \mathbf{0} && (19) \\ \mathbf{k}_{n+1}(0) &= \sum_i \pi_i \mathbf{k}_n(i). && (20) \end{aligned}$$

Note that, because Focal has at most one mother, grandmother, etc., the expected number of mothers, grandmothers, etc. is also the probability of having a living mother, grandmother, etc.

2.1.3 Sisters and nieces

The sisters of Focal, and their children, who are the nieces of Focal, form the first set of side branches in the kinship network of Figure 1. Following Goodman, Keyfitz, and Pullum (1974), it is convenient to divide the sisters of Focal into older and younger sisters, because they follow different dynamics.

$\mathbf{m}(x)$ **= older sisters of Focal.** Once Focal is born, she accumulates no more older sisters, so the subsidy term is $\boldsymbol{\beta}(x) = \mathbf{0}$. At Focal's birth, her older sisters are the children $\mathbf{a}(i)$ of the mother of Focal at the age i of Focal's mother at Focal's birth. This age is unknown, so the initial condition $\mathbf{m}_0$ is a mixture of the age distributions of children with mixing distribution $\boldsymbol{\pi}$.

$$\begin{aligned} \mathbf{m}(x+1) &= \mathbf{U}\mathbf{m}(x) + \mathbf{0} && (21) \\ \mathbf{m}_0 &= \sum_i \pi_i \mathbf{a}(i). && (22) \end{aligned}$$

$\mathbf{n}(x)$ **= younger sisters of Focal.** Focal has no younger sisters when she is born, so the initial condition is $\mathbf{n}_0 = \mathbf{0}$. Younger sisters are produced by reproduction of Focal's mother, so the subsidy term is the reproduction of the mothers at age x of Focal.

$$\begin{aligned} \mathbf{n}(x+1) &= \mathbf{U}\mathbf{n}(x) + \mathbf{F}\mathbf{d}(x) && (23) \\ \mathbf{n}_0 &= \mathbf{0}. && (24) \end{aligned}$$

$\mathbf{p}(x)$ **= nieces through older sisters of Focal.** At the birth of Focal, these nieces are the granddaughters of the mother of Focal, so the initial condition is mixture of granddaughters with mixing distribution $\boldsymbol{\pi}$. New nieces through older sisters are the result of reproduction by the older sisters, at age x, of Focal.

$$\mathbf{p}(x+1) = \mathbf{U}\mathbf{p}(x) + \mathbf{F}\mathbf{m}(x) \tag{25}$$
$$\mathbf{p}_0 = \sum_i \pi_i \mathbf{b}(i). \tag{26}$$

$\mathbf{q}(x)$ **= nieces through younger sisters of Focal.** At the birth of Focal she has no younger sisters, and hence has no nieces through these sisters. Thus the initial condition is $\mathbf{q}_0 = \mathbf{0}$. New nieces are produced by reproduction of the younger sisters of Focal.

$$\mathbf{q}(x+1) = \mathbf{U}\mathbf{q}(x) + \mathbf{F}\mathbf{n}(x) \tag{27}$$
$$\mathbf{q}_0 = \mathbf{0}. \tag{28}$$

2.1.4 Aunts and cousins

Aunts and cousins form another level of side branching on the kinship network; their dynamics follow the same principles as those for sisters and nieces.

$\mathbf{r}(x)$ **= aunts older than mother of Focal.** These are the older sisters of the mother of Focal. Once Focal is born, her mother accumulates no new older sisters, so the subsidy term is $\boldsymbol{\beta}(x) = \mathbf{0}$. The initial age distribution of these aunts, at the birth of Focal, is a mixture of the age distributions $\mathbf{m}$ of older sisters, with mixing distribution $\boldsymbol{\pi}$

$$\mathbf{r}(x+1) = \mathbf{U}\mathbf{r}(x) + \mathbf{0} \tag{29}$$
$$\mathbf{r}_0 = \sum_i \pi_i \mathbf{m}(i). \tag{30}$$

$\mathbf{s}(x)$ **= aunts younger than mother of Focal.** These are the younger sisters of the mother of Focal. These aunts are the children of the grandmother of Focal, and thus the subsidy term comes from reproduction by the grandmothers of Focal. The initial age distribution of these aunts, at the birth of Focal, is a mixture of the age distributions $\mathbf{n}$ of younger sisters, with mixing distribution $\boldsymbol{\pi}$.

$$\mathbf{s}(x+1) = \mathbf{U}\mathbf{s}(x) + \mathbf{F}\mathbf{g}(x) \tag{31}$$
$$\mathbf{s}_0 = \sum_i \pi_i \mathbf{n}(i). \tag{32}$$

$\mathbf{t}(x)$ **= cousins from aunts older than mother of Focal.** These are the children of the older sisters of the mother of Focal, and thus the nieces of the mother of Focal through her older sisters. The subsidy term comes from reproduction by the older sisters of the mother of Focal.The initial condition is a mixture of the age distributions of nieces through older sisters, with mixing distribution $\boldsymbol{\pi}$.

$$\begin{aligned} \mathbf{t}(x+1) &= \mathbf{U}\mathbf{t}(x) + \mathbf{F}\mathbf{r}(x) && (33) \\ \mathbf{t}_0 &= \sum_i \pi_i \mathbf{p}(i). && (34) \end{aligned}$$

$\mathbf{v}(x)$ **= cousins from aunts younger than mother of Focal.** These are the nieces of the mother of Focal through her younger sisters. The subsidy term comes from reproduction by the younger sisters of the mother of Focal. The initial condition is a mixture of the age distributions of nieces through younger sisters, with mixing distribution $\boldsymbol{\pi}$.

$$\begin{aligned} \mathbf{v}(x+1) &= \mathbf{U}\mathbf{v}(x) + \mathbf{F}\mathbf{s}(x) && (35) \\ \mathbf{v}_0 &= \sum_i \pi_i \mathbf{q}(i). && (36) \end{aligned}$$

2.1.5 Model summary

The dynamics of the entire network of 14 types of consanguineal kin in Figure 1 are summarized in Table 1. Note that each kin type depends only on kin types above it in the table. Thus there are no circular dependencies to render the model insoluble. Note also that the side chains through nieces, cousins, etc. can be extended just as the chains of descendants and ancestors are extended in equations (12) and (19).

Table 1: Summary of the components of the kin model given in equations (4) and (5)

Symbol	Kin	Initial condition	Subsidy $\boldsymbol{\beta}(x)$
a	daughters	$\mathbf{0}$	$\mathbf{Fe}_x$
b	granddaughters	$\mathbf{0}$	$\mathbf{Fa}(x)$
c	great-granddaughters	$\mathbf{0}$	$\mathbf{Fb}(x)$
d	mothers	$\boldsymbol{\pi}$	$\mathbf{0}$
g	grandmothers	$\sum_i \pi_i \mathbf{d}(i)$	$\mathbf{0}$
h	great-grandmothers	$\sum_i \pi_i \mathbf{g}(i)$	$\mathbf{0}$
m	older sisters	$\sum_i \pi_i \mathbf{a}(i)$	$\mathbf{0}$
n	younger sisters	$\mathbf{0}$	$\mathbf{Fd}(x)$
p	nieces via older sisters	$\sum_i \pi_i \mathbf{b}(i)$	$\mathbf{Fm}(x)$
q	nieces via younger sisters	$\mathbf{0}$	$\mathbf{Fn}(x)$
r	aunts older than mother	$\sum_i \pi_i \mathbf{m}(i)$	$\mathbf{0}$
s	aunts younger than mother	$\sum_i \pi_i \mathbf{n}(i)$	$\mathbf{Fg}(x)$
t	cousins from aunts older than mother	$\sum_i \pi_i \mathbf{p}(i)$	$\mathbf{Fr}(x)$
v	cousins from aunts younger than mother	$\sum_i \pi_i \mathbf{q}(i)$	$\mathbf{Fs}(x)$

3. Derived properties of kin

Because the model provides the age distributions of all types of kin, it makes it possible to compute what might be called derived properties of the age distribution of kin. These might be linear functions of the age distribution, leading to a model

$$\mathbf{k}(x+1) = \mathbf{U}\mathbf{k}(x) + \boldsymbol{\beta}(x) \tag{37}$$

$$\mathbf{k}(0) = \mathbf{k}_0 \tag{38}$$

$$\mathbf{y}(x) = \boldsymbol{\Psi}(x)\mathbf{k}(x) \tag{39}$$

where $\mathbf{y}(x)$ is a vector of the property in question at age x of focal, and $\boldsymbol{\Psi}(x)$ is the matrix of a linear transformation from the age distribution to the property vector. Examples of such derived properties include

1. Numbers of kin, in which case $\boldsymbol{\Psi}(x) = \mathbf{1}_\omega^{\mathsf{T}}$.
2. Prevalence, in which case $\boldsymbol{\Psi}(x)$ is a vector containing, e.g., age-specific prevalence of some condition, such as disease, disability, health, labor force participation, etc.
3. Measures of economic dependency. For example, if three dependency categories are defined (young-age dependency, old-age dependency, and independence), then each row of $\boldsymbol{\Psi}$ would pick out the ages corresponding to one of the dependency groups. For six age classes, with two classes in each dependency category, the resulting matrix would be

$$\mathbf{\Psi} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{40}$$

4. Coresidence probability. This is actually a special case of prevalence, where the condition is "coresiding with Focal."

Nonlinear functions of $\mathbf{k}(x)$ (e.g., dependency ratios) can also be calculated. One important set of such derived properties are the mean, and other moments, of the age of a particular set of relatives.

5. Moments of age distribution. Define vectors

$$\mathbf{c}_i = \begin{pmatrix} 0.5^i & 1.5^i & \cdots & (\omega - 0.5)^i \end{pmatrix}^{\mathsf{T}} \qquad i = 1, 2, \ldots. \tag{41}$$

   Define μ_i as the ith moment of age (so that the mean age is μ_1). The ith moment of the age of the kin $\mathbf{k}(x)$ is

$$\mu_i(x) = \mathbf{c}_i^{\mathsf{T}} \frac{\mathbf{k}(x)}{\|\mathbf{k}(x)\|} \tag{42}$$

   (provided, of course, that $\|\mathbf{k}(x)\| > 0$). In particular, the mean and variance of the age of kin are

$$E\left(\mu(x)\right) = \mu_1(x) \tag{43}$$

$$V\left(\mu(x)\right) = \mu_2(x) - \mu_1(x)^2. \tag{44}$$

A useful operation is the aggregation of kin types. It is possible to aggregate the kinship network in Figure 1 by adding the appropriate vectors.

6. Aggregation of kin.
   Figure 1 disaggregates the older and younger sisters of Focal. The total number of sisters is the sum of the older and younger sisters,

$$\text{sisters} = \mathbf{m}(x) + \mathbf{n}(x). \tag{45}$$

   An important aggregation is that based on degree. Degrees of kinship are defined in both civil and religious law, and determine ability to marry, aspects of inheritance, jury selection, restrictions on nepotism in hiring, and other fascinating things. According to one version,

$$\text{first degree kin} = \mathbf{a}(x) + \mathbf{d}(x) \tag{46}$$

$$\text{second degree kin} = \mathbf{b}(x) + \mathbf{g}(x) + \mathbf{m}(x) + \mathbf{n}(x) \tag{47}$$

$$\text{third degree kin} = \mathbf{h}(x) + \mathbf{c}(x) + \mathbf{r}(x) + \mathbf{s}(x) + \mathbf{p}(x) + \mathbf{q}(x). \tag{48}$$

4. Death of kin

The experience of the death of close relatives can have long-lasting effects on an individual (e.g., Umberson et al. 2017). The experience by Focal of the death of kin can be calculated directly from the kinship model. To do so, we expand the kin population vector $\mathbf{k}$ to include dead as well as living kin, creating a new vector

$$\tilde{\mathbf{k}} = \left(\frac{\mathbf{k}_{\text{living}}}{\mathbf{k}_{\text{dead}}} \right). \tag{49}$$

The tilde distinguishes this multistate vector from the vector containing only living relatives.

Two possibilities present themselves for calculations with deceased relatives. We can calculate the deaths of kin experienced by Focal at a given age x, or the cumulative deaths experienced by Focal up to a given age x. The calculations require only a simple change to the matrices $\mathbf{U}$ and $\mathbf{F}$, and the vector $\mathbf{k}_0$, in order to account for both living and dead kin.

In order for $\mathbf{k}_{\text{dead}}(x)$ to capture the age distribution of the deaths experienced by Focal at age x, $\mathbf{U}$ is replaced by the block-structured matrix

$$\tilde{\mathbf{U}} = \left(\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \hline \mathbf{M} & \mathbf{0} \end{array} \right). \tag{50}$$

The mortality matrix $\mathbf{M}$ contains the transition probabilities from ages of kin (columns of $\mathbf{M}$) to the state of being dead at a particular age (rows of $\mathbf{M}$). Thus

$$\mathbf{M} = \mathcal{D}(\mathbf{q}). \tag{51}$$

The matrix $\mathbf{0}$ in the lower right corner of $\tilde{\mathbf{U}}$ removes the dead individuals after a single time step. The result is the projection

$$\tilde{\mathbf{k}}(x+1) = \tilde{\mathbf{U}}\tilde{\mathbf{k}}(x) + \tilde{\boldsymbol{\beta}}(x). \tag{52}$$

The fertility matrix $\mathbf{F}$ that appears in $\boldsymbol{\beta}(x)$ is replaced by the matrix

$$\tilde{\mathbf{F}} = \left(\begin{array}{c|c} \mathbf{F} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right) \tag{53}$$

which asserts no dead offspring are produced (this could be modified to account for stillbirth) and that the dead do not reproduce.

To calculate the cumulative deaths experienced by Focal up to age x, rather than the deaths experienced at a given age, the matrix $\mathbf{U}$ is replaced by

$$\tilde{\mathbf{U}} = \left(\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \hline \mathbf{M} & \mathbf{I} \end{array} \right) \tag{54}$$

where again

$$\mathbf{M} = \mathcal{D}(\mathbf{q}).$$

The identity matrix in the lower right corner of $\tilde{\mathbf{U}}$ keeps the dead kin in an absorbing state corresponding to their age at death.

The initial condition $\tilde{\mathbf{k}}_0$ for the partitioned kin vector accounts for the fact that Focal has experienced no deaths at the time of her birth. Thus,

$$\tilde{\mathbf{k}}_0 = \left(\begin{array}{c} \mathbf{k}_0 \\ \hline \mathbf{0} \end{array} \right) \tag{55}$$

where $\mathbf{k}_0$ is the initial vector for kin $\mathbf{k}$ as described in Table 1.

These calculations can be extended to include deaths that occur before the birth of Focal (e.g., "your grandmother died before you were born") or after the death of Focal (e.g., Queen Victoria died in 1901 at the age of 81, but of her 87 great-grandchildren, several were born after 1901, and of course other descendants continue to appear). These extensions will be presented elsewhere.

5. An example: Changes in the kinship network of Japan

As an example of the model, I explore the implications for the kinship network of changes in the mortality and fertility schedules of Japanese women from 1947 and 2014. This period saw dramatic changes in both mortality (life expectancy increased by about 60%) and fertility (total fertility rate decreased by 70% and the net reproductive rate declined by about 60%), as shown in Figure 2.

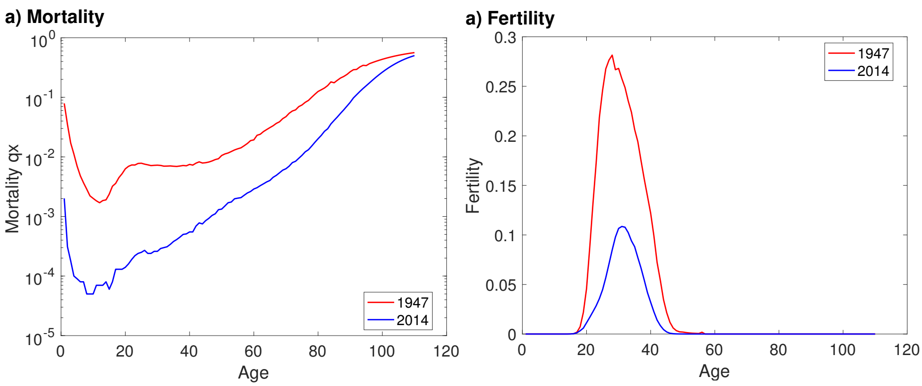
	1947	2014	% change
life exp	54	87	+61%
TFR	4.6	1.4	−70%
R_0	1.7	0.7	−59%

The matrices $\mathbf{U}$ and $\mathbf{F}$ are created from the mortality (q_x) schedules and the age-specific fertility schedules from the Human Mortality Database and Human Fertility Database (Human Mortality Database 2018; Human Fertility Database 2018). MATLAB code for the calculations is given in the online materials.

Note that this is just an example; it is not intended as a detailed examination of the kinship demography of Japan. Also note that for convenience I will speak of, e.g., "Japan in 1947" instead of the more correct "a stable population subject to the period mortality and fertility schedules of Japan as measured in 1947."

For the convenience of the reader, results of the calculations are collected together, in graphical form, for selected types of kin, in Section 7. For the truly curious, an Online Supplementary collection contains figures for all types of kin for each of the categories examined here.

Figure 2: Mortality and fertility. The mortality and fertility schedules for Japanese women in 1947 and 2014



Source: Data from Human Mortality Database (2018) and Human Fertility Database (2018).

5.1 Age distributions

Figure 4 shows the age distributions of mothers, grandmothers, daughters, granddaughters, sisters, and cousins, for a Focal individual aged 30 and aged 70. The mothers of Focal at 30 are slightly older under 2014 rates than under 1947 rates, and far more common. Focal at age 70 has essentially no chance of a living mother in 1947, but still some chance of a very elderly living mother in 2014 (Figure 4a). The situation with grandmothers is similar (Figure 4b), but more extreme. No living grandmothers remain at age 70 of Focal, but at age 30 grandmothers are about 4 times more likely and about 10 years older in 2014 compared to 1947.

Daughters and granddaughters (Figures 4c and d) are less abundant in 2014 than in 1947, reflecting the lower fertility in 2014. Granddaughters are more abundant than daughters in 1947, but less abundant in 2014, reflecting the net reproductive rates at those two times (population increase in 1947, population decline in 2014).

The age distributions of sisters and cousins (Figure 4e and f) show the effects of the mortality difference between 1947 and 2014. In 1947, Focal loses about 40% of her sisters and cousins between the ages of 30 and 70. In 2014, there is almost no loss of sisters or cousins between these ages.

5.2 Numbers of kin

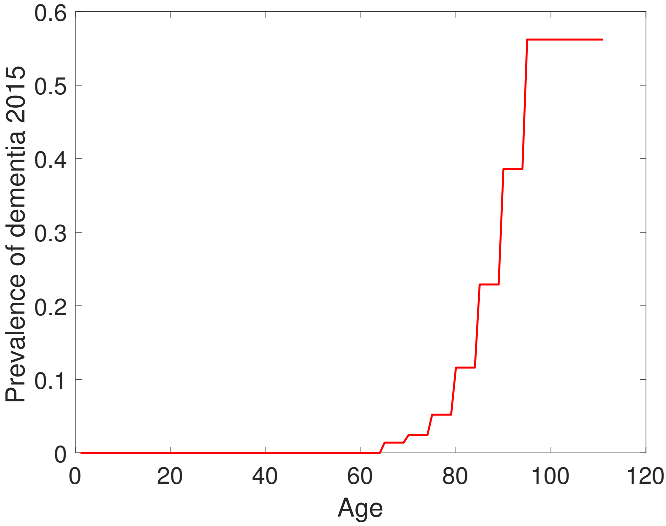
Figure 5 shows the numbers of living kin as a function of the age of Focal. Comparing daughters, granddaughters, and great-granddaughters (Figures 5a, c, and e) shows the integrated effects of mortality and fertility changes between 1947 and 2014. In 1947, Focal reaches a peak of about 3 times more daughters than does Focal in 2014, but the number of living daughters declines after about age 40 of Focal. In 2014, fewer daughters are produced, and there is hardly any decline in the number of daughters due to mortality. Comparing the numbers of granddaughters and great-granddaughters shows the pattern hinted at in Figure 4: Focal in 1947 has progressively more descendants in each generation, while Focal in 2014 has fewer.

For ancestors (Figures 5b, d, and f), the intergenerational pattern is reversed. Focal in 2014 is more likely to have a surviving mother than Focal in 1947; the differential increases for grandmothers and great-grandmothers.

5.3 Prevalence of dementia

As an example of using equation (39) to map from age distributions to the prevalence of some condition, consider kin suffering from dementia. Figure 3 shows the age-specific prevalence of dementia in Japanese females in 2015 (Fukawa 2018): a roughly exponential increase starting at age 60. In the absence of information on the prevalence pattern in 1947, I will use this prevalence schedule for both years.

Figure 3: Dementia prevalence. Age-specific prevalence of dementia among Japanese women in 2015



Source: Data from Fukawa (2018).

Figure 6 shows the numbers of kin with dementia, as a function of the age of Focal, in 1947 and 2014. Focal is far more likely to have a mother, grandmother, or great-grandmother with dementia in 2014 than in 1947 (Figures 6a, c, and d). The difference is large (about 7-fold for mothers, even greater for grandmothers and great-grandmothers). The same holds for sisters (Figure 6b) and aunts (Figure 4d). Among cousins, the difference is not as great, but the prevalence of dementia among kin is still higher in 2014 than 1947.

5.4 Mean and variance of ages of kin

The means and standard deviations of the ages of several types of kin are shown in Figures 7 and 8. Mean ages naturally increase with the age of Focal. For both ancestors (mothers, grandmothers, etc.) and descendants (daughters, granddaughters, etc.) there is little difference between 1947 and 2014, perhaps because the timing of fertility does not change much between those years.

The standard deviation of descendants increases with age of Focal, and is slightly higher under 1947 rates than 2014 rates, presumably because of the higher mortality rates in 1947. The standard deviation of the age of ancestors decreases with the age of

Focal, with no consistent differences between 1947 and 2014 rates. Maximum standard deviations are on the order of 6 to 8 years. Differences between 1947 and 2014 rates are small relative to other properties, because the timing of reproduction shows only minor changes.

5.5 Dependency of kin

Figure 9 shows, as a function of the age of Focal, the numbers of kin in three categories of dependence. Young dependence is defined here as ages 0–15, old dependence as ages greater than 65, and independence as ages 16–65. These could easily be replaced with more detailed descriptors of economic contribution.

Figure 9 shows results for 1947 in solid lines, and 2014 in dashed lines. Dependent children, grandchildren, and great-grandchildren accumulate earlier, and much more rapidly, for Focal in 1947 than in 2014. Focal in 1947 was much more likely to have dependent great-granddaughters than in 2014, reflecting the greater numbers of descendants under those conditions (cf. Figure 5).

The pattern is reversed when considering dependent mothers, grandmothers, and great-grandmothers, which are much more abundant in 2014 than in 1947. A short description of the pattern would be that Focal in 1947 confronts more dependent children and descendants, but in 2014 she is faced with more dependent parents and ancestors.

5.6 Death of kin

Turning now to the death of kin, Figure 10 shows the experience of death of kin at each age of Focal, and Figure 11 shows the cumulative deaths experienced up to each age of Focal. As far as deaths of kin are concerned, the world changed dramatically between 1947 and 2014. The deaths of daughters, granddaughters, mothers, sisters, and aunts occur earlier and far more frequently under the rates of 1947. Focal in 2014 will almost never experience the death of a daughter or granddaughter (Figures 10a, b; 11a and b). It is rare for Focal in 2014 to experience the death of a sister before the age of 60, but in 1947 such deaths occur frequently from the birth of Focal.

6. Discussion

The model of Goodman, Keyfitz, and Pullum (1974) relies on multiple integrals to calculate expected numbers of kin of different kinds, at a specified age of a focal individual. The method presented here, in contrast, is a coupled system of matrix equations that projects the population of kin forward as Focal ages. The mathematics (formally, a coupled system of non-autonomous matrix difference equations) may sound more complicated. It is not. As with any dynamical system, the dynamic equations carry out the necessary integrations, but with much more flexibility. Together, the assumptions of homogeneity and time invariance make it possible to extend the equations for parents and children to include all the kin shown in Table 1, and even beyond that, as in equation (12) for arbitrary levels of descendants. A brief comparison of the results given by Goodman, Keyfitz, and Pullum (1974) and those produced by this model shows qualitative agreement, but with quantitative differences probably due to the (unspecified) choice of numerical integration methods applied to the coarsely-resolved (5 year age intervals) life tables available in 1974. The freedom from the need to carry out such numerical integration, and from the error propagation involved with multiple integrals, is a strength of the present method.

One advantage of formal mathematical specification is that it makes explicit the assumptions underlying an analysis. As Goodman, Keyfitz, and Pullum (1974) pointed out repeatedly, these results are not expected to give the same results as a census of the kin of individuals of different ages, precisely because the assumptions are counterfactuals. The value of comparing calculated kinship structures with empirical kinship censuses is not to test the mathematics, but to see how the actual kinship network is warped by violation of the assumptions.

It will be interesting to relax the assumptions. Relaxing the assumption of homogeneity will require extending the state space to include additional dimensions affecting kinship (marital status is one obvious possibility) in age$\times$stage or multistate models (Caswell et al. 2018). Parity dependence is another important dimension. Schoen (2019) presents theory for close kin in terms of parity progression, under the assumption that all women live to the end of their reproductive years and that mortality does not affect children. He emphasizes that parity progression, when used as a model for fertility, automatically captures some important aspects of sibship and family formation. Incorporating age and parity into the reproductive component of the model here will permit exploration of these effects under less restrictive assumptions.

The analysis here, and the example in Section 5, are formulated in terms of female survival and fertility, and relatives through the female line. It is clearly possible to carry out the same analysis using male survival and fertility; it will be interesting to do so to see the effect of the extended timing of male fertility, especially in hunter–gatherer populations (e.g., Tuljapurkar, Puleston, and Gurven 2007). A generalization to include both male and female kin, through both male and female lines of descent, will be presented elsewhere.

In addition to extensions to male as well as female kin, several other extensions are under active investigation. The present model is age-classified, which implies that age alone determines mortality and fertility. Stage-classified and multistate models will allow age to interact with other characteristics (marital status, health status, etc.). Relaxing the assumption of time invariance will require the extension of the time domain to include not only the age x of Focal but also the time before or after the birth of Focal.

Finally, note that the results of these calculations, like those of Goodman, Keyfitz, and Pullum (1974), provide *expected* age distributions. While the kin of Focal form a population, that population is small and thus subject to demographic stochasticity. Stochastic versions of the model could be constructed using branching process methods, as discussed by Pullum (1982). Connections of multitype branching processes to matrix population models are explored by Pollard (1966), Caswell (2001), and Caswell and Vindenes (2018). Alternatively, stochastic realizations of the dynamic models here, or even complete microsimulation models (e.g., Wachter 1997), can provide information on variances and higher moments.

The analysis, presented here as an example, using vital rates for Japan shows how this method can reveal differences in the kinship patterns implied by different mortality and fertility schedules. The differences, using rates in 1947 and 2014, are dramatic. In 1947, the kinship structure of a Japanese woman was full of the experience of the death of close kin, often at young ages. In 2014, such experiences are rare or non-existent. On the other hand, a Japanese woman in 2014 is many times more likely to experience elderly dependent kin, or kin suffering from dementia, than was the case under 1947 rates. These results are presented here as examples of the use of the kinship theory presented here, but they make it obvious that using the theory to explore the effects of changes in mortality and fertility is an important next step.

7. Figures

Figure 4: Age distributions. The age distributions of several types of kin, at ages 30 (solid lines) and 70 (dashed lines) of Focal. Calculated from the vital rates of Japan in 1947 (red) and 2014 (blue).

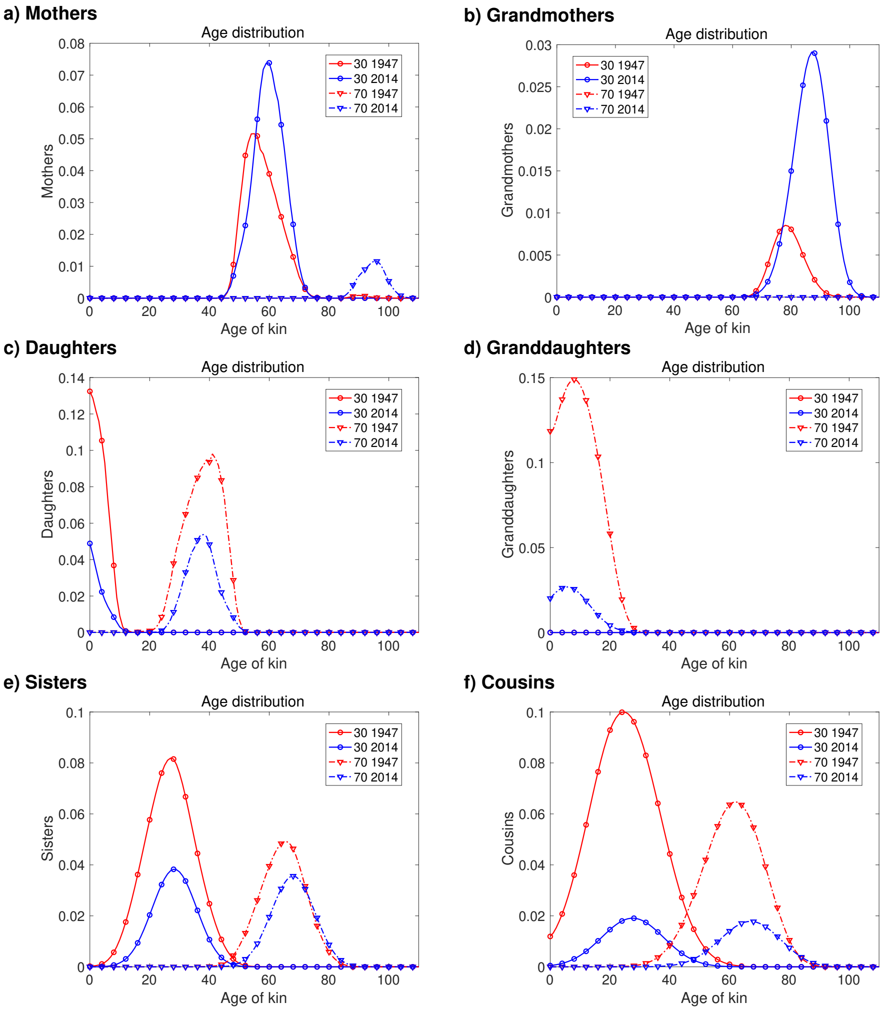


Figure 5: Numbers. Numbers of kin of several types, as a function of the age of Focal. Calculated from the vital rates of Japan in 1947 (red) and 2014 (blue).

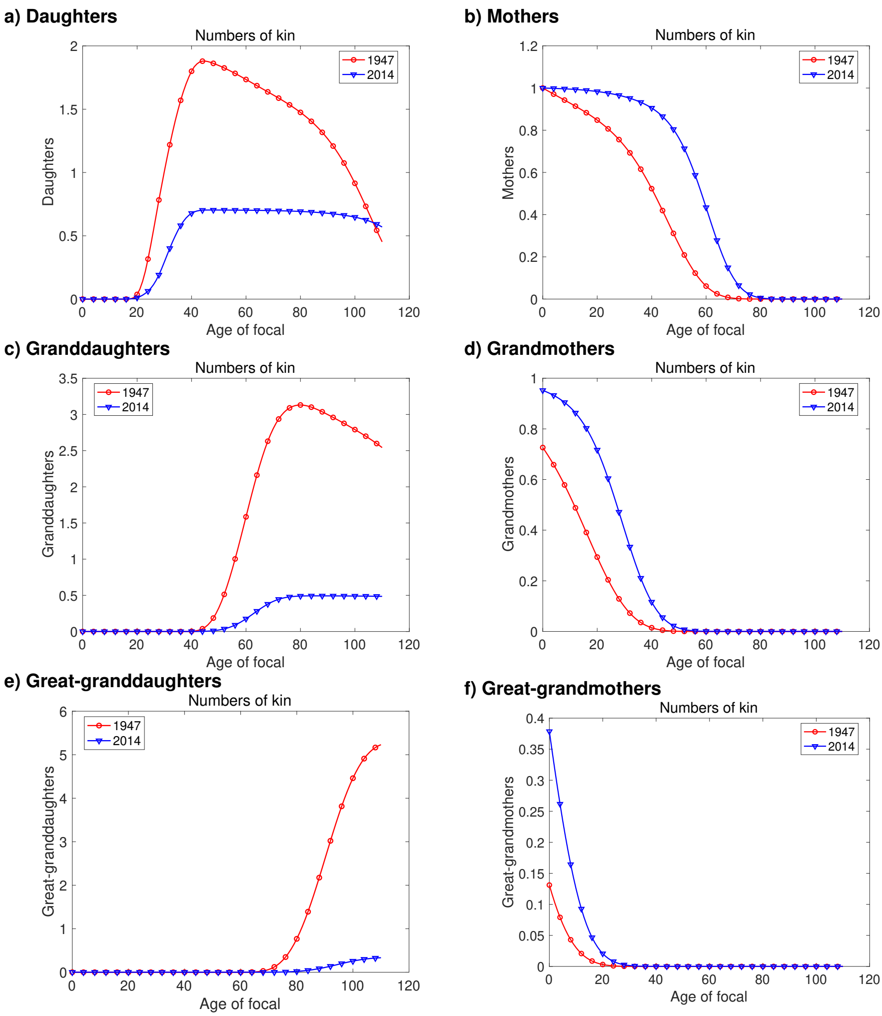


Figure 6: Kin with dementia. Numbers of kin of several types suffering from dementia, as a function of the age of Focal. Calculated from the vital rates of Japan in 1947 (red) and 2014 (blue), using dementia prevalence rates for Japanese females in 2015.

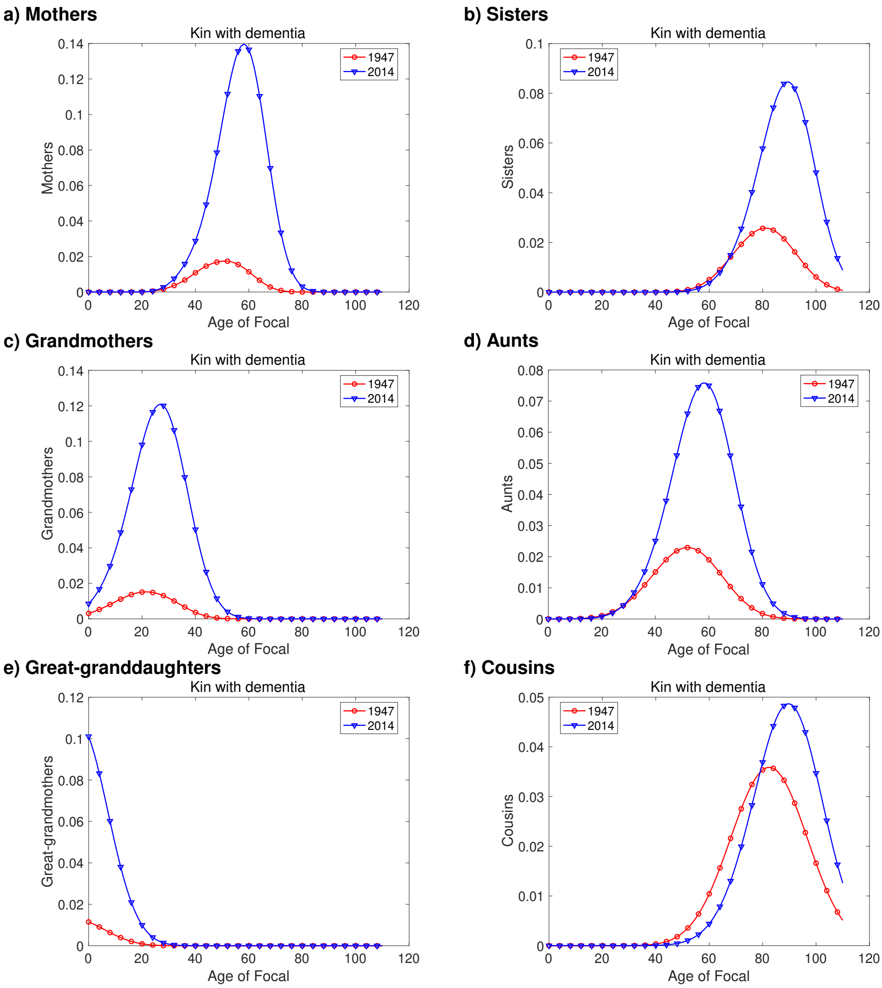


Figure 7: Mean age. The mean age of kin of several types, as a function of the age of Focal. Calculated from the vital rates of Japan in 1947 (red) and 2014 (blue). The mean age is set to zero when the number of kin drops below 10^{-9}.

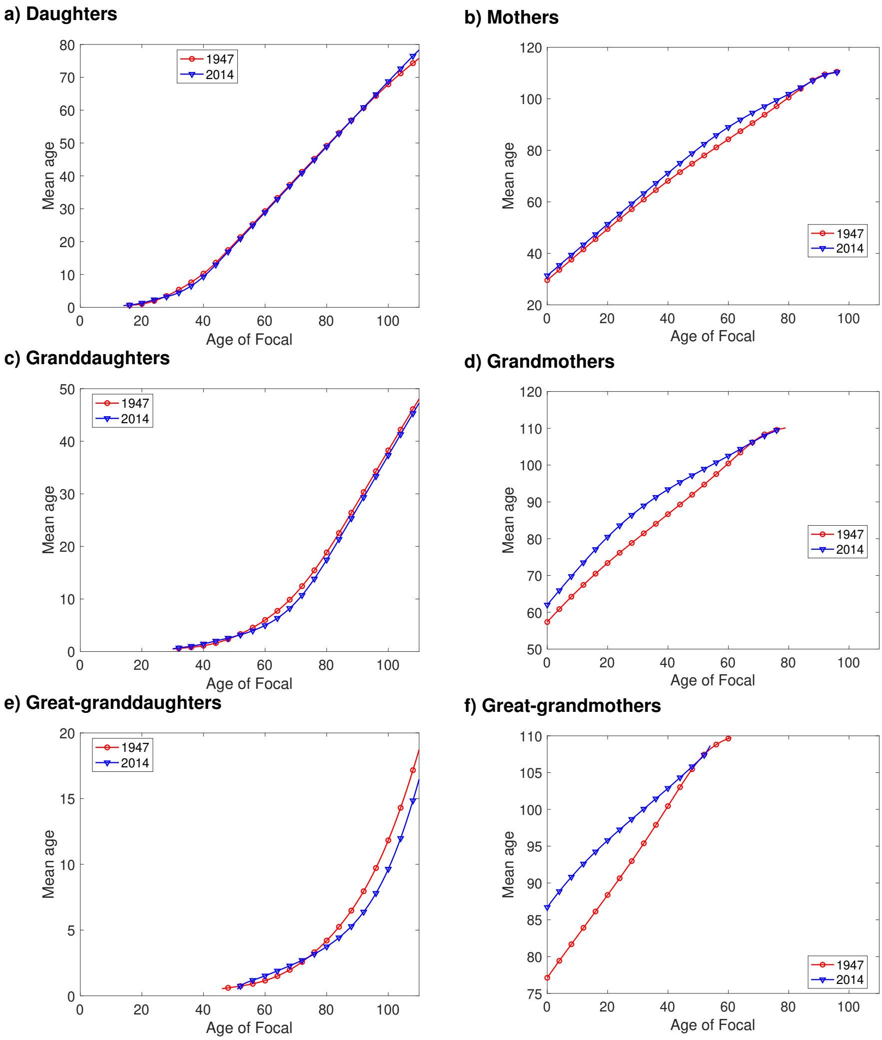


Figure 8: Standard deviation of age. The standard deviation (in years) of the age of kin of several types, as a function of the age of Focal. Calculated from the vital rates of Japan in 1947 (red) and 2014 (blue).

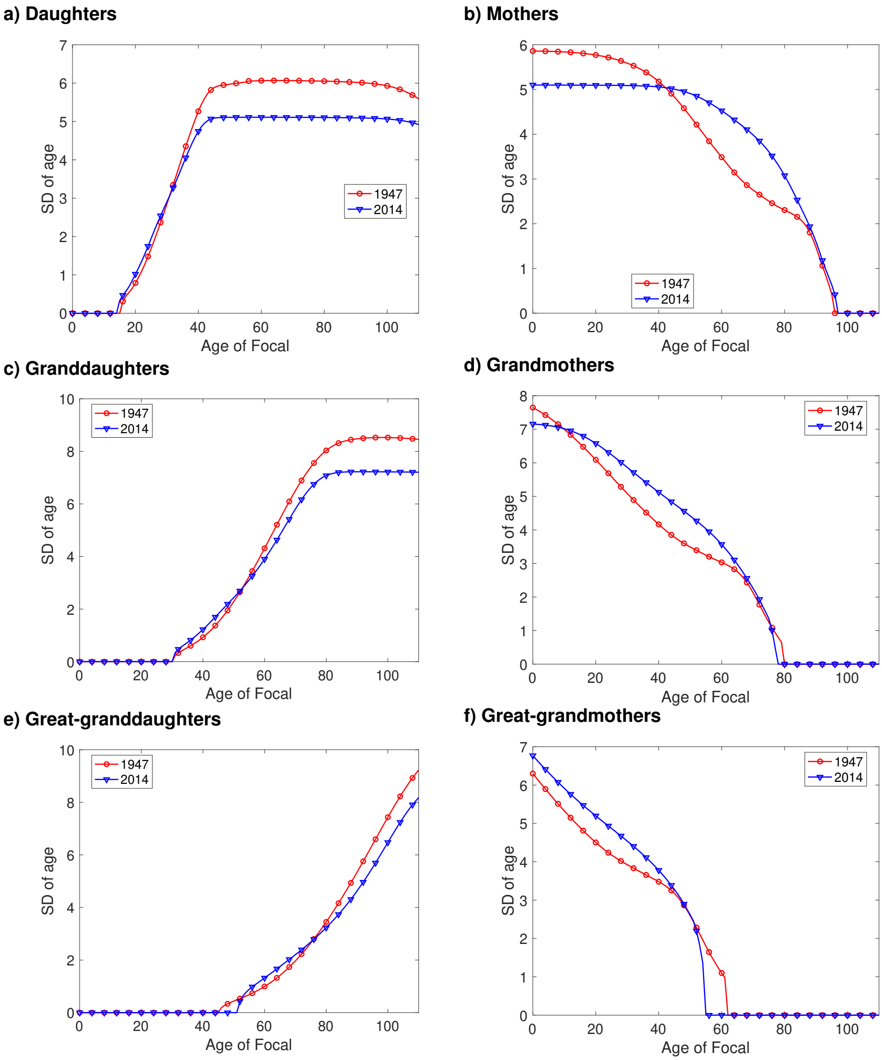


Figure 9: Dependency of kin. Numbers of kin, of several types, in three different dependency categories: young dependents aged 0–16, old dependents aged more than 65, and independent kin aged 16–65, as a function of the age of Focal. Calculated from the vital rates of Japan in 1947 (solid lines) and 2014 (dashed lines).

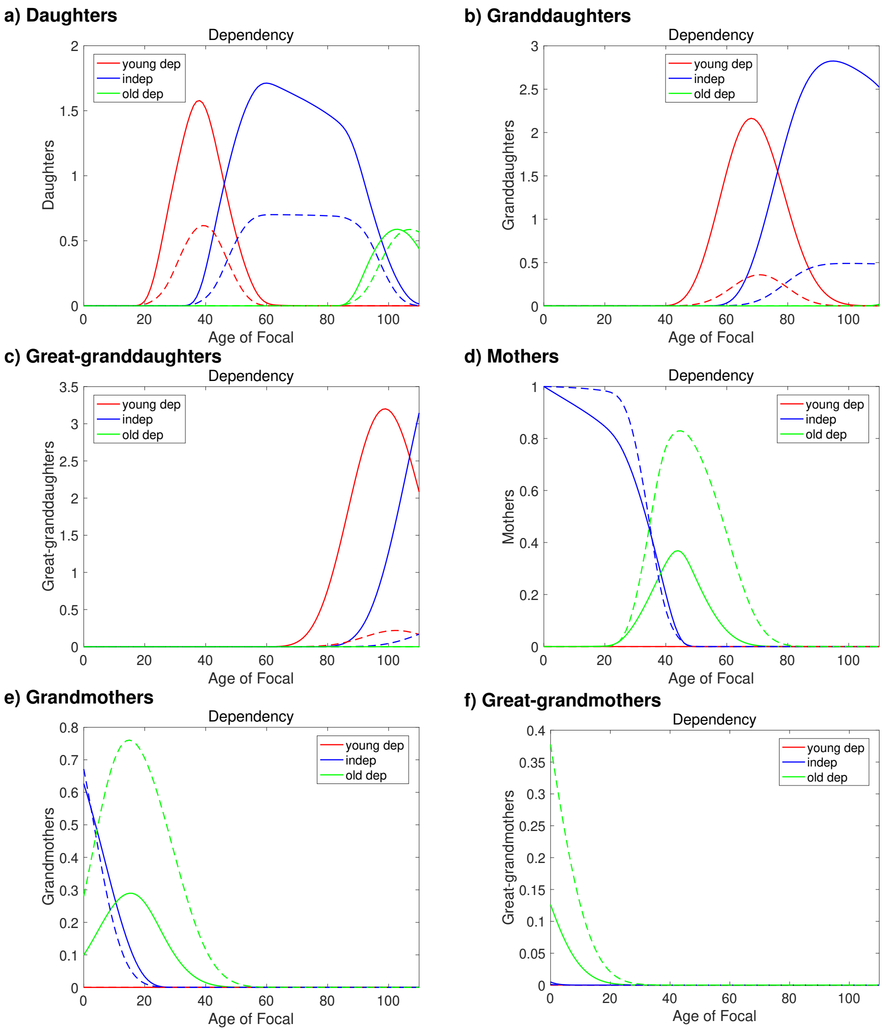


Figure 10: Experienced deaths. Numbers of deaths of kin, of several types, experienced by Focal at each age. Calculated from the vital rates of Japan in 1947 and 2014.

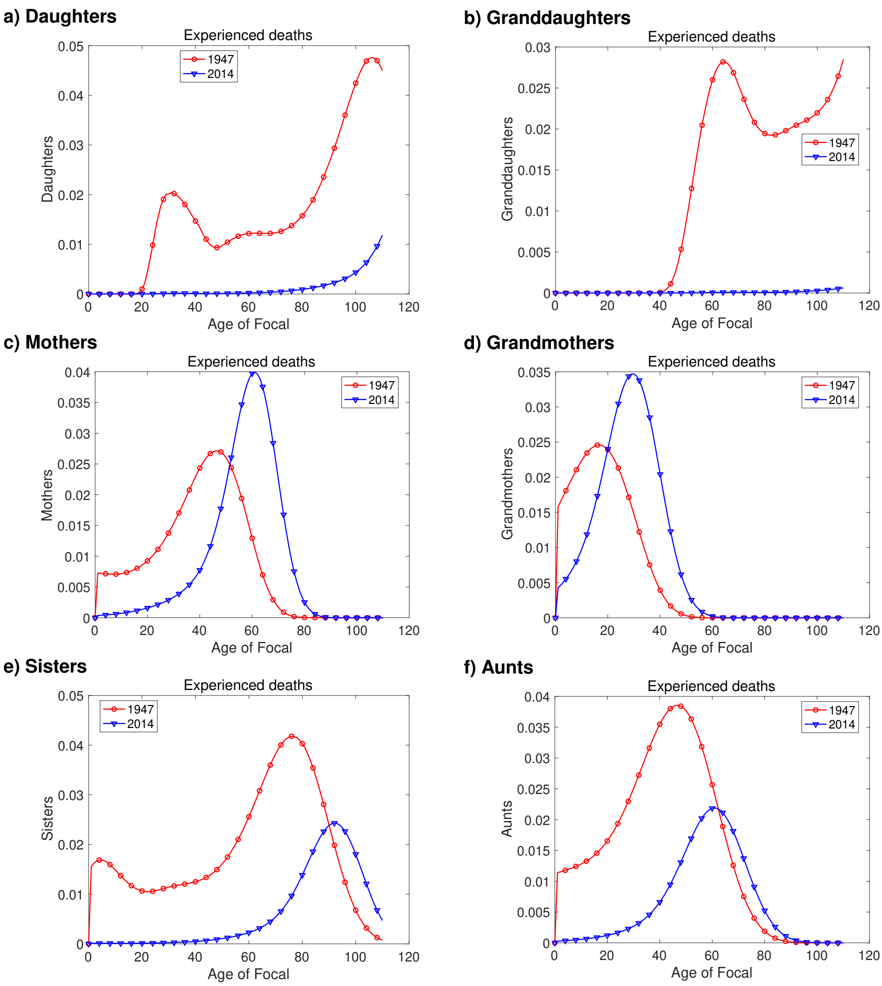
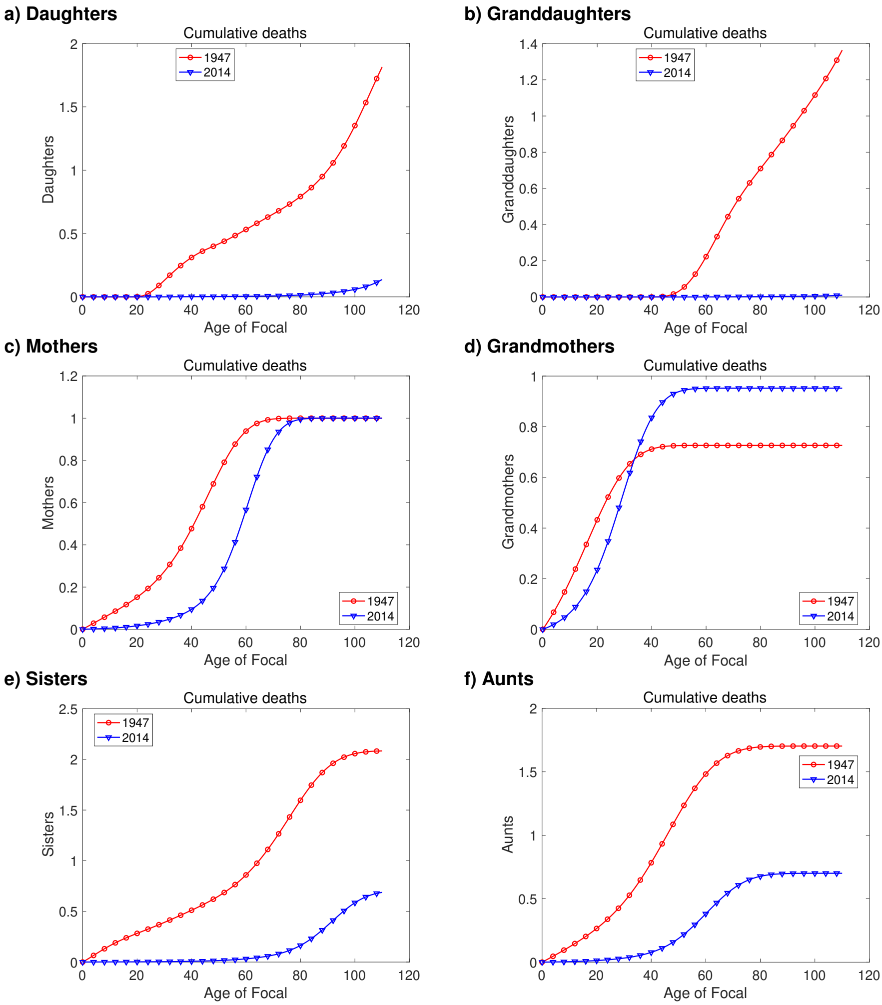


Figure 11: Cumulative deaths. The cumulative numbers of deaths of kin experienced by Focal up to each age. Calculated from the vital rates of Japan in 1947 and 2014.



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The formal demography of kinship II : Multistate models of kinship*

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Contents

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Abstract

Recent formal demographic models of kinship have focused on the age distribution of kin, as a function of the age of the focal individual. We present a multistate extension of the kinship model that permits classification by age and other (generically, 'stage') criteria. The analysis uses the vec-permutation matrix approach, creating a block-structured vector to describe the age×stage distribution of the population, and block-structured matrices that project that population taking into account the age and stage dependency of survival, stage transitions, and age transitions. The resulting matrix formulation is directly comparable to the age-classified model of Caswell (2019). As one important case of age×stage -classification, we derive the dynamics of the kinship network of the focal individual in terms of age and parity. The results provide the age×paritydistribution of all types of kin, at all ages of the focal individual. As an example, we apply the model to the population of Slovakia from 1960 to 2014, using data from the Human Fertility Database and the Human Mortality Database. The results show the results, in terms of the kinship network, of reductions in fertility and of the transition to high parity states.

1 Introduction

Caswell (2019) presented a formal demographic model of the kinship network that would result from a specified mortality and fertility schedule. That model produces the expected age distribution, of any kind of kin, at any age of the Focal individual. The structure of the model explicitly assumes that age is the only factor governing the production and disappearance of kin, so that the necessary ingredients for the analysis are age schedules of mortality and fertility. It also assumes that the only interesting property of the kin is age, or something that can be calculated as a function of age.

The model of Caswell 2019 is a good starting point, but the assumption of age specificity is a weakness. It is easy to think of other characteristics, in addition to age, that influence survival and fertility, and thus the dynamics of kin. In this paper we extend the age-specific kinship model to incorporate such additional factors. We refer to these generically as multistate or age-stage models.[1] After presenting the general multistate kinship framework, we apply it to the case of maternal parity.

Parity (the number of children that a women has had up to a given age) influences fertility, and probably mortality, although parity-specific survival schedules are not easily obtained. In addition, incorporating parity into the analysis provides extra information about family structures and kinship (Schoen, 2019b,a).

2 A brief review of the age-specific kinship model

As in Caswell (2019), and following Goodman, Keyfitz, and Pullum (1974), kin are defined relative to a focal individual named (for purposes of reference) Focal. In the age-classified model, Focal is a member of a populatiiion in which all individuals are subject to the same age schedules of mortality and fertility. These are captured in survival and fertility matrices; e.g., for the case of four age classes,

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & [p4] \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

[1]Models that incorporate three or more characteristics have been called hyperstate models (Roth and Caswell, 2016). They generalize the construction to be developed here, and should apply to kinship models as well, but we do not consider them further here.

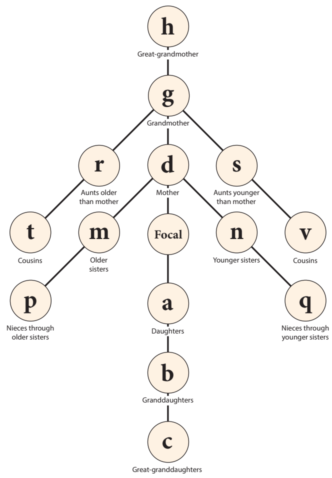

Figure 1: The network of kin defined in Goodman, Keyfitz, and Pullum (1974) and Keyfitz and Caswell (2005), with symbols ($\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$, etc.) used to denote the age distribution vectors of each type of kin of Focal.

where p_i is the probability of survival and f_i the effective fertility of age class i. It is assumed that the rates are time-invariant and have been in effect long enough that the stable age distribution can be used to calculate the age distribution of mothers. That age distribution is given by the right eigenvector $\mathbf{w}$ of $\mathbf{A} = \mathbf{U} + \mathbf{F}$ corresponding to the dominant eigenvalue λ, normalized to sum to 1.

The kin considered in the calculation are defined in Figure 1. The age-specific analysis treated the kin of any type as a populatioon, described by an age distribution vector denoted by the letters in Figure 1 ($\mathbf{a}$ for daughters, $\mathbf{b}$ for granddaughters, etc.). For a generic kin $\mathbf{k}(x)$ the dynamics were written as

$$
\begin{aligned}
\mathbf{k}(x+1) &= \mathbf{U}\mathbf{k}(x) + \boldsymbol{\beta}(x) && (2)\\
\mathbf{k}(0) &= \mathbf{k}_0 && (3)
\end{aligned}
$$

where x is the age of Focal. The model tracks the development of the network of kin surrounding Focal as she ages. The term $\mathbf{U}\mathbf{k}(x)$ applies the survival matrix to the age distribution of kin at age x of Focal .he term $\boldsymbol{\beta}(x)$ is the reproductive subsidy, the production of ne kin by some other type of kin (e.g., granddaughters are produced by the reproduction of daughters). The initial condition $\mathbf{k}_0$ specifies the age distribution of the kin at the birth of Focal. For example, $\mathbf{a}_0 = m\mathbf{b}_0 = \mathbf{c}_0$ because we may be quite sure that Focal has no daughter, granddaughters, or great-granddaughters at her birth.

It is certain that Focal has one mother alive at the time of her birth, but the age of the mother is unknown. However, the distribution of the ages of mothers in the stable population is given by

$$
\boldsymbol{\pi} = \frac{\mathbf{F}(1,:)^{\mathsf{T}} \circ \mathbf{w}}{\|\mathbf{F}(1,:)^{\mathsf{T}} \circ \mathbf{w}\|} \tag{4}
$$

where $\mathbf{F}(1,:)$ is the first row of $\mathbf{F}$ and $\|\cdot\|$ denotes the 1-norm. At Focal's birth, it was assumed that her mother is a randomly selected individual from this distribution, so that

$$
\mathbf{d}_0 = \boldsymbol{\pi}. \tag{5}
$$

Applying this procedure methodically to all the kin in Figure 1, Caswell (2019) derived the projection models for all kin; the results are given in Table 1.

Table 1: Summary of the components of the age-specific kin model of Caswell (2019). Compare with the age×stage -specific model summarized in Table 2. Reproduced under the terms of a CC-BY license.

Symbol	Kin	initial condition $\mathbf{k}_0$	Subsidy $\boldsymbol{\beta}(x)$
$\mathbf{a}$	daughters	$\mathbf{0}$	$\mathbf{F}\mathbf{e}_x$
$\mathbf{b}$	granddaughters	$\mathbf{0}$	$\mathbf{F}\mathbf{a}(x)$
$\mathbf{c}$	great-granddaughters	$\mathbf{0}$	$\mathbf{F}\mathbf{b}(x)$
$\mathbf{d}$	mothers	$\boldsymbol{\pi}$	$\mathbf{0}$
$\mathbf{g}$	grandmothers	$\sum_i \pi_i \mathbf{d}(i)$	$\mathbf{0}$
$\mathbf{h}$	great-grandmothers	$\sum_i \pi_i \mathbf{g}(i)$	$\mathbf{0}$
$\mathbf{m}$	older sisters	$\sum_i \pi_i \mathbf{a}(i)$	$\mathbf{0}$
$\mathbf{n}$	younger sisters	$\mathbf{0}$	$\mathbf{F}\mathbf{d}(x)$
$\mathbf{p}$	nieces via older sisters	$\sum_i \pi_i \mathbf{b}(i)$	$\mathbf{F}\mathbf{m}(x)$
$\mathbf{q}$	nieces via younger sisters	$\mathbf{0}$	$\mathbf{F}\mathbf{n}(x)$
$\mathbf{r}$	aunts older than mother	$\sum_i \pi_i \mathbf{m}(i)$	$\mathbf{0}$
$\mathbf{s}$	aunts younger than mother	$\sum_i \pi_i \mathbf{n}(i)$	$\mathbf{F}\mathbf{g}(x)$
$\mathbf{t}$	cousins from aunts older than mother	$\sum_i \pi_i \mathbf{p}(i)$	$\mathbf{F}\mathbf{r}(x)$
$\mathbf{v}$	cousins from aunts younger than mother	$\sum_i \pi_i \mathbf{q}(i)$	$\mathbf{F}\mathbf{s}(x)$

3 The vec-permutation model for multistate demography

We turn now to the multistate generalization of the age-specific kin model. We develop the model carefully, because it is critical in order to arrive at the end result.

The multistate model classifies individuals by jointly by age and some other characteristic, referred to generically as "stage." In Section 5, the stage variable will be parity. The model is constructred using the vec-permutation matrix approach introduced by Hunter and Caswell (2005) and described in detail in Caswell et al. (2018); it has been applied, inter alia, to frailty (Caswell, 2014; Hartemink, Missov, and Caswell, 2017), epidemiology (Klepac and Caswell, 2011) and genetics (de Vries and Caswell, 2019).

Suppose the population has ω age classes and s stages, and let k_{ij} denote the number of individuals, of some type of kin, in stage i and age class j. The state of the kin population can be described by the array

$$\mathcal{K} = \begin{pmatrix} k_{11} & \cdots & k_{1\omega} \\ \vdots & & \vdots \\ k_{s1} & \cdots & k_{s\omega} \end{pmatrix} \tag{6}$$

from which a population vector is obtained as

$$\tilde{\mathbf{k}} = \operatorname{vec} \mathcal{K} = \left(\begin{array}{c} k_{11} \\ \vdots \\ k_{s1} \\ \hline \vdots \\ \hline k_{1\omega} \\ \vdots \\ k_{s\omega} \end{array} \right) \tag{7}$$

The following sets of matrices, together, define the age- and stage-dependent demographic rates.

$$\begin{aligned}
\mathbf{U}_i &= \text{stage transitions for age class } i \qquad i = 1, \ldots, \omega && (8)\\
\mathbf{D}_j &= \text{age advancement for stage } j \qquad j = 1, \ldots, s && (9)\\
\mathbf{F}_i &= \text{stage-specific fertility for age class } i \qquad i = 1, \ldots, \omega && (10)\\
\mathbf{H}_j &= \text{offspring assignment for stage } j \qquad j = 1, \ldots, s && (11)
\end{aligned}$$

The matrices $\mathbf{U}_i$ and $\mathbf{F}_i$ are of dimension $s \times s$. The matrices $\mathbf{D}_j$ and $\mathbf{H}_j$ are of dimension $\omega \times \omega$.

The entries in $\mathbf{U}_i$ are transition probabilities among stages. Mortality may or may not be included in $\mathbf{U}_i$. If it is not included, the entries are transition probabilities conditional on survival, and $\mathbf{U}_i$ is column-stochastic. If mortality is included, $\mathbf{U}_i$ is column sub-stochastic.

The matrix $\mathbf{D}_j$ advances individuals of stage j from one age class to the next. If mortality is accounted for in the $\mathbf{U}_i$, then $\mathbf{D}_j$ is a matrix with ones on the subdiagonal and zeros elsewhere. Otherwise, $\mathbf{D}_j$ contains age-specific survival probabilities for stage j on the subdiagonal and zeros elsewhere.

The matrix $\mathbf{F}_i$ captures stage-specific fertility; the (k, ℓ) entry is the per capita production of stage k offspring by stage ℓ individuals, in age class i. The matrix $\mathbf{H}_j$ assigns the offspring of individuals in stage j to the appropriate age class. If offspring are born into the first age class, then $\mathbf{H}_j$ contains ones in the first row and zeros elsewhere.

Use these matrices to construct block diagonal matrices; e.g., from the $\mathbf{U}_i$, construct

$$\mathbb{U} = \begin{pmatrix} \mathbf{U}_1 & & \\ & \ddots & \\ & & \mathbf{U}_\omega \end{pmatrix} \tag{12}$$

with a similar construction for $\mathbb{D}$, $\mathbb{F}$, and $\mathbb{H}$. These matrices are all of dimension $s\omega \times s\omega$. Finally, construct matrices $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{F}}$ as

$$\begin{aligned}
\tilde{\mathbf{U}} &= \mathbf{K}_{s,\omega}^{\mathsf{T}} \, \mathbb{D} \, \mathbf{K}_{s,\omega} \, \mathbb{U} && (13)\\
\tilde{\mathbf{F}} &= \mathbf{K}_{s,\omega}^{\mathsf{T}} \, \mathbb{H} \, \mathbf{K}_{s,\omega} \, \mathbb{F} && (14)
\end{aligned}$$

where $\mathbf{K}_{s,\omega}$ is the vec-permutation matrix appropriate to the $s \times \omega$ array $\mathcal{N}$ (Henderson and Searle, 1981). The matrix $\tilde{\mathbf{U}}$ captures survival, transitions, and aging of extant individuals as they move among age-stage categories. The matrix $\tilde{\mathbf{F}}$ captures the production of new individuals by fertility, and the assignment of those newborn individuals into appropriate age-stage categories. They play exactly the role in the multistate model as the matrices $\mathbf{U}$ and $\mathbf{F}$ in Caswell (2019).

Various simplifications occur depending on whether individual stage categories are fixed or dynamic. If the stages are fixed, as for example birth weight or mother's age at birth, then the $\mathbf{U}_i$ are diagonal matrices. If stages are dynamic, as for example parity or martial status, then the structure of the $\mathbf{U}_i$ reflects the possible transitions as these develop over age.

The structure of the $\mathbf{F}_i$ depends on how the stage of the mother affects the stage of the offspring. If all offspring are born into the same stage, as for example with parity, then $\mathbf{F}_i$ will contain non-zero entries only in the row corresponding to that stage. If offspring can be born into multiple stages, as for example when stages are defined as maternal age at birth, then the pattern of entries in the $\mathbf{F}_i$ will reflect this transmission.

4 Multistate kinship calculations

4.1 Population structure and age distribution

The population projection matrix is given by $\tilde{\mathbf{A}} = \tilde{\mathbf{U}} + \tilde{\mathbf{F}}$. We retain the assumption that Focal is a member of a population with the stable age-stage structure implied by $\tilde{\mathbf{A}}$. This stable structure is given by the right eigenvector $\tilde{\mathbf{w}}$ (scaled to sum to 1) of $\tilde{\mathbf{A}}$ corresponding to its largest eigenvalue.

The joint age×stage distribution of mothers in the stable population is

$$\tilde{\boldsymbol{\pi}} = \frac{\left(\mathbf{1}_{s\omega}^{\mathsf{T}}\tilde{\mathbf{F}}\right)^{\mathsf{T}} \circ \tilde{\mathbf{w}}}{\left\|\left(\mathbf{1}_{s\omega}^{\mathsf{T}}\tilde{\mathbf{F}}\right)^{\mathsf{T}} \circ \tilde{\mathbf{w}}\right\|} \qquad s\omega \times 1 \tag{15}$$

The marginal age distribution of mothers in the stable population is

$$\boldsymbol{\pi}^{\text{age}} = \left(\mathbf{I}_{\omega} \otimes \mathbf{1}_{s}^{\mathsf{T}}\right)\tilde{\boldsymbol{\pi}} \qquad s \times 1 \tag{16}$$

The multistate model admits the possibility that reproduction may produce more than one type of offspring; i.e., producing offspring that appear in different stages. These must be combined somehow in order to define the age of mothers of "offspring." The term $\mathbf{1}_{s\omega}^{\mathsf{T}}\tilde{\mathbf{F}}$ in equation (15) simply adds them up. One could insgtead combine them with some kind of weighted sum. I do not consider this further here.

Given the multistate model, the dynamics of the population $\mathbf{k}$ of some type of kin, jointly classified by age and stage, are given by

$$\tilde{\mathbf{k}}(x+1) = \tilde{\mathbf{U}}\tilde{\mathbf{k}}(x) + \tilde{\boldsymbol{\beta}}(x) \tag{17}$$

with initial condition

$$\tilde{\mathbf{k}}(0) = \tilde{\mathbf{k}}_0. \tag{18}$$

Our construction has reduced the multistate model to the same mathematical form[2] as the age-classified model. In the process it shows how to incorporate age- and stage-specific rates into the projection of $\tilde{\mathbf{k}}$. The matrix formulation permits us to use much of the same model structure as in the age-classified model.

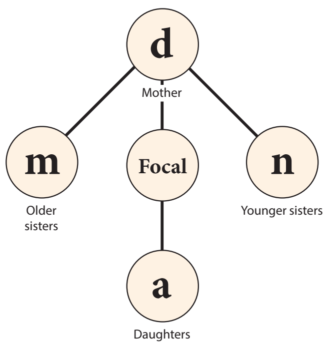

Figure 2: The mother-daughter-sister core of the kinship network in Figure 1

[2] The joke about the physicist, the engineer, and the mathematician confronting a fire in a wastebasket is left as an exercise for the reader.

4.2 The core kinship network

The core of the kinship network is comprised of Focal, her mother, her daughters, her older sisters, and her younger sisters, as shown in Figure 2. The model for the entire kinship network (Figure 1) can be derived from this core (Caswell, 2019). Schoen (2019a) presented a similar core, differing only in that it did not distinguish older and younger sisters as is done here. The following sections will derive the dynamics of each component of the core network.

4.2.1 The dynamics of Focal

In the age-classified model, the dynamics of Focal were trivial. Focal was an individual female assumed to be alive at age x. Thus the age distribution of Focal at age x was $\mathbf{e}_x$, a unit vector of length ω with a 1 in the xth place and zeros elsewhere. No further attention was needed.

In the multistate model, Focal is again assumed to be alive at age x, but is described by a joint age×stage distribution. This age×stage distribution will change as Focal ages and moves among stages. Thus our analysis begins with the dynamics of Focal, in essence treating her as one of her own relatives. We define

$$\tilde{\boldsymbol{\phi}}(x) = \text{age}\times\text{stage distribution of Focal at age } x \tag{19}$$

The dynamics of $\tilde{\boldsymbol{\phi}}(x)$ are obtained by writing $\tilde{\mathbf{U}}$ as

$$\tilde{\mathbf{U}} = \tilde{\mathbf{G}}\,\tilde{\boldsymbol{\Sigma}} \tag{20}$$

where

$$\tilde{\boldsymbol{\Sigma}} = \mathcal{D}(\tilde{\boldsymbol{\sigma}}) \tag{21}$$

and $\tilde{\boldsymbol{\sigma}} = \mathbf{1}^{\mathsf{T}}_{s\omega}\tilde{\mathbf{U}}$ is the vector of survival probabilities, The matrix $\tilde{\mathbf{G}}$ contains transition proababilities conditional on survival. Then, the conditional age×stage distribution of Focal at age x satisfies

$$\begin{aligned} \tilde{\boldsymbol{\phi}}(x+1) &= \tilde{\mathbf{G}}\tilde{\boldsymbol{\phi}}(x) + \mathbf{0} && (22)\\ \tilde{\boldsymbol{\phi}}(0) &= \tilde{\boldsymbol{\phi}}_0. && (23) \end{aligned}$$

The initial condition $\tilde{\boldsymbol{\phi}}_0$ is the joint age×stage distribution of children, at birth, in the stable population,

$$\tilde{\boldsymbol{\phi}}_0 = \frac{\tilde{\mathbf{F}}\tilde{\mathbf{w}}}{\left\|\tilde{\mathbf{F}}\tilde{\mathbf{w}}\right\|}. \tag{24}$$

4.2.2 Daughters of Focal

Daughters are the result of the reproduction of Focal. Since Focal is assumed to be alive at age x, the subsidy vector for daughters is $\tilde{\boldsymbol{\beta}}(x) = \tilde{\mathbf{F}}\tilde{\boldsymbol{\phi}}(x)$, where $\tilde{\boldsymbol{\phi}}(x)$ is the age×stage vector for Focal at age x. Because we may be sure that Focal has no daughters when she is born, the initial condition is $\tilde{\mathbf{a}}_0 = \mathbf{0}$. Thus

$$\begin{aligned} \tilde{\mathbf{a}}(x+1) &= \tilde{\mathbf{U}}\tilde{\mathbf{a}}(x) + \tilde{\mathbf{F}}\tilde{\boldsymbol{\phi}}(x) && (25)\\ \tilde{\mathbf{a}}_0 &= \mathbf{0}. && (26) \end{aligned}$$

4.2.3 Mothers of Focal

The population $\tilde{\mathbf{d}}(x)$ of mothers at age x of Focal consists of at most a single individual (step-mothers are not considered here) Because no new mothers arrive after Focal's birth, the subsidy term is $\tilde{\boldsymbol{\beta}}(x) = \mathbf{0}$.

The initial condition for the population of mothers requires additional steps not familiar from the age-classified model. Recall that the age×stage distribution of the mothers of new children in the stable population is calculated as $\tilde{\boldsymbol{\pi}}$ in (15). The marginal distribution of the ages of these mothers is $\boldsymbol{\pi}^{\text{age}}$, from (16).

In the age-classified model (Caswell, 2019), the initial population $\mathbf{d}_0$ of the mothers was a mixture of unit vectors $\mathbf{e}_i$ (a vector of length ω with a 1 in the ith entry and zeros elsewhere), with mixing distribution $\boldsymbol{\pi}$. In the multistate version, the mother of Focal is still known to be alive at the birth of Focal, but in addition, her stage distribution must be accounted for. This distribution may be subject to additional constraints beyond the requirement that Focal's mother be alive at her birth; e.g., after the birth of Focal, her mother cannot be in parity state 0 (see Section 5).

Let $\tilde{\mathbf{z}}(i)$ be the age×stage distribution vector for a mother of Focal at age i, satisfying whatever constraints are appropriate. Then

$$\tilde{\mathbf{d}}_0 = \sum_i \pi_i^{\text{age}}\, \tilde{\mathbf{z}}(i). \tag{27}$$

Thus the model for the population of mothers is

$$\begin{aligned} \tilde{\mathbf{d}}(x+1) &= \tilde{\mathbf{U}}\tilde{\mathbf{d}}(x) + \mathbf{0} & (28)\\ \tilde{\mathbf{d}}(0) &= \tilde{\mathbf{d}}_0 & (29) \end{aligned}$$

4.2.4 Older sisters of Focal

Once Focal is born, she accumulates no more older sisters, so the subsidy term is $\tilde{\boldsymbol{\beta}}(x) = \mathbf{0}$. At Focal's birth, her older sisters are the children $\tilde{\mathbf{a}}(i)$ of the mother of Focal at the age i of Focal's mother at Focal's birth. This age is unknown, so the initial condition $\tilde{\mathbf{m}}_0$ is a mixture of the age distributions of children with mixing distribution given by the marginal age distribution of mothers $\boldsymbol{\pi}^{\text{age}}$.

$$\begin{aligned} \tilde{\mathbf{m}}(x+1) &= \tilde{\mathbf{U}}\tilde{\mathbf{m}}(x) + \mathbf{0} & (30)\\ \tilde{\mathbf{m}}_0 &= \sum_i \boldsymbol{\pi}_i^{\text{age}}\, \tilde{\mathbf{a}}(i). & (31) \end{aligned}$$

4.2.5 Younger sisters of Focal

Focal has no younger sisters when she is born, so the initial condition is $\mathbf{n}_0 = \mathbf{0}$. Younger sisters are produced by reproduction of Focal's mother, so the subsidy term is the reproduction of the mothers at age x of Focal.

$$\begin{aligned} \tilde{\mathbf{n}}(x+1) &= \tilde{\mathbf{U}}\tilde{\mathbf{n}}(x) + \tilde{\mathbf{F}}\tilde{\mathbf{d}}(x) & (32)\\ \tilde{\mathbf{n}}_0 &= \mathbf{0}. & (33) \end{aligned}$$

4.3 From the core to the rest of the kinship network

The dynamics of the age×stage distribution of daughters, mothers, and sisters of Focal are the same as the dynamics of the age distributions in the age-classified model, only replacing the matrices $\mathbf{U}$ and $\mathbf{F}$ with $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{F}}$ and modifying the initial conditions appropriately. The extension of the core kinship network to the entire network (granddaughters, nieces, cousins,

Table 2: Summary of the components of the age×stage -classified kinship model. Matrices and vectors bearing tildes (e.g., $\tilde{\mathbf{a}}$) inherit the age×stage structure given by equation (7). Compare with the age-classified summary in Table 1. The vector $\boldsymbol{\pi}^{\text{age}}$ is the marginal distribution of the ages of mothers in the stable population. The vector $\tilde{\mathbf{z}}(i)$ is the expected age×stage distribution of mothers of age i at the birth of Focal.

Symbol	Kin	initial condition $\mathbf{k}_0$	Subsidy $\boldsymbol{\beta}(x)$
$\tilde{\boldsymbol{\phi}}$	Focal	$\tilde{\boldsymbol{\phi}}_0$	$\mathbf{0}$
$\tilde{\mathbf{a}}$	daughters	$\mathbf{0}$	$\tilde{\mathbf{F}}\tilde{\boldsymbol{\phi}}(x)$
$\tilde{\mathbf{b}}$	granddaughters	$\mathbf{0}$	$\tilde{\mathbf{F}}\tilde{\mathbf{a}}(x)$
$\tilde{\mathbf{c}}$	great-granddaughters	$\mathbf{0}$	$\tilde{\mathbf{F}}\tilde{\mathbf{b}}(x)$
$\tilde{\mathbf{d}}$	mothers	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{z}}(i)$	$\mathbf{0}$
$\tilde{\mathbf{g}}$	grandmothers	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{d}}(i)$	$\mathbf{0}$
$\tilde{\mathbf{h}}$	great-grandmothers	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{g}}(i)$	$\mathbf{0}$
$\tilde{\mathbf{m}}$	older sisters	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{a}}(i)$	$\mathbf{0}$
$\tilde{\mathbf{n}}$	younger sisters	$\mathbf{0}$	$\tilde{\mathbf{F}}\tilde{\mathbf{d}}(i)$
$\tilde{\mathbf{p}}$	nieces via older sisters	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{b}}(i)$	$\tilde{\mathbf{F}}\tilde{\mathbf{m}}(x)$
$\tilde{\mathbf{q}}$	nieces via younger sisters	$\mathbf{0}$	$\tilde{\mathbf{F}}\tilde{\mathbf{n}}(i)$
$\tilde{\mathbf{r}}$	aunts older than mother	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{m}}(x)$	$\mathbf{0}$
$\tilde{\mathbf{s}}$	aunts younger than mother	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{n}}(i)$	$\tilde{\mathbf{F}}\tilde{\mathbf{g}}(x)$
$\tilde{\mathbf{t}}$	cousins: aunts older than mother	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{p}}(i)$	$\tilde{\mathbf{F}}\tilde{\mathbf{r}}(x)$
$\tilde{\mathbf{v}}$	cousins: aunts younger than mother	$\sum_i \boldsymbol{\pi}_i^{\text{age}}\tilde{\mathbf{q}}(i)$	$\tilde{\mathbf{F}}\tilde{\mathbf{s}}(x)$

etc.) follows closely the derivations as presented in Caswell (2019). The results are shown in Table 2; comparison with Table 1 shows how similar the results are, once the age×stage model is formulated using the vec permutation matrix. For the curious, the complete set of derivations is given in Appendix **??**. As in the age-classified model, each kin type depends only on kin types above it in the table. Thus there are no circular dependencies to render the model insoluble. Note also that the side chains through nieces, cousins, etc. can be extended just as the chains of descendants and ancestors are extended in Caswell (2019).

5 Multistate kinship: Age and parity

Parity (the number of live births that a female has had) is a particularly interesting stage variable. Parity affects reproductive decisions and fertility, and age-specific fertility rates are an aggregation of age×parity-specific rates. Individuals move through parity stages over time; a first birth is a transition from parity 0 to parity 1, a second birth a transition from parity 1 to parity 2, etc. Parity also has effects on mortality (e.g., Barclay and Kolk, 2019), but in the absence of age×parity-specific mortality data, we do not consider this effect here. However, if data were available, mortality effects could be included in the multistate model (as part of the age advancement matrices $\mathbf{D}_j$ in equation (13)).

The initial condition for Focal's mother, at the time of Focal's birth, must satisfy three conditions: Focal has exactly one mother, Focal's mother is alive, and Focal's mother, being a mother, is not in parity class 0. Only the first two conditions applied in the age-classified model (Caswell, 2019), and so the initial vector for Focal's mother was a mixture, over the distribution of mother's ages, of unit vectors. Now, in the age×parity-classified model, the initial vector is a mixture, over the marginal distribution of mother's ages, of age×parity vectors with parity 0 removed, normalized to sum to 1. To construct this initial vector $\tilde{\mathbf{d}}_0$, define the vectors

$$\mathbf{z}_i = \frac{\left[\mathbf{e}_i^{\mathsf{T}} \otimes (\mathbf{I}_s - \mathbf{E}_{11})\right] \tilde{\mathbf{w}}}{\left\| \left[\mathbf{e}_i^{\mathsf{T}} \otimes (\mathbf{I}_s - \mathbf{E}_{11})\right] \tilde{\mathbf{w}} \right\|} \qquad i = 1, \ldots, s \tag{34}$$

where $\mathbf{e}_i$ is the ith unit vector of length ω and $\mathbf{E}_{11}$ is an $s \times s$ matrix with a one in the $(1,1)$ position and zeros elsewhere. Then

$$\tilde{\mathbf{d}}_0 = \sum_i \pi_i^{\text{age}} \, \mathbf{z}_i. \tag{35}$$

Because backwards parity transitions are impossible, these conditions on $\tilde{\mathbf{d}}_0$ also guarantee that $\tilde{\mathbf{d}}(x)$ at later ages will not contain parity 0.

Similarly, Focal's grandmother and great-grandmother (and further generations if included) must not have parity 0 after the birth of Focal. Because the initial condition for grandmothers is a mixture of vectors for the mothers,

$$\tilde{\mathbf{g}}_0 = \sum_i \pi_i^{\text{age}} \, \tilde{\mathbf{d}}(i), \tag{36}$$

equation (35) guarantees that this constraint will be satisfied. The same argument applies for great-grandmothers.

5.1 Age and parity: matrix construction

The matrices describing kin dynamics for the age×paritymodel are constructed as follows. Suppose (as is the case for data in the HFD) that six parity classes are identified (0,1,2,3,4,5+). The parity transition matrix for age class x is

$$\mathbf{U}_x = \begin{pmatrix} 1-u_1 & 0 & 0 & 0 & 0 & 0 \\ u_1 & 1-u_2 & 0 & 0 & 0 & 0 \\ 0 & u_2 & 1-u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 1-u_4 & 0 & 0 \\ 0 & 0 & 0 & u_4 & 1-u_5 & 0 \\ 0 & 0 & 0 & 0 & u_5 & 1 \end{pmatrix} (x) \tag{37}$$

where $u_i(x)$ is the probabiity of an ith birth to a woman of age x and parity $i-1$. The parity class 5+ contains women of parity 5 and all higher parities. The transition probabilities are obtained, as per Jasilioniene et al. (2019, p. 51), from conditional parity-specific birth rates mi as

$$u_i = \frac{mi}{1 + (1-0.5)mi} \qquad i = 1, 2, 3, 4, 5+ \tag{38}$$

These rates are, of course, functions of age, yielding matrices $\mathbf{U}_x$ for $x = 1, \ldots, \omega$,

The age advancement matrices $\mathbf{D}_j$ are all equal and given by (e.g., for $\omega = 4$),

$$\mathbf{D}_j = \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \end{pmatrix} \qquad j = 1, \ldots, s \tag{39}$$

where $p_i = 1 - q_i$ is the probability of survival of an individual in age class i. If age-specific mortality was known to differ among parity classes (this relationship seems to be complicated; e.g., Barclay and Kolk 2019; Sonneveldt, Plosky, and Stover 2013), the $\mathbf{D}_j$ would differ among parity classes.

In an age×paritymodel, reproduction is associated with transitions among parity classes. The probabilities of those transitions are included in the $\mathbf{U}$ matrices, so that

$$\mathbf{F}_x = 0.5 \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} (x) \tag{40}$$

Entries appear only in the first row because all offspring are born into parity 0. The factor 0.5 assumes an even sex ratio at birth. Because all offspring are born into the first age class, the age assignment matrices $\mathbf{H}_j$ are the same for all parity classes; e.g., for $\omega = 4$

$$\mathbf{H}_j = \mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad j = 1, \ldots, s \tag{41}$$

5.2 Age and parity for Slovakia

As an example of the potential of age×paritykinship analysis, we show some results using data from the Human Fertility and Human Mortality databases. The Human Fertility Database (Human Fertility Database, 2019) contains data on age- and age×parity-specific fertility for, as of 2019, thirty-one countries. The file of conditional age-specific fertility rates contains, for each age, the probability of a first, second, third, fourth, and fifth or higher birth. These are the probabilities of transition from parity 0 to 1, from 1 to 2, from 2 to 3, from 3 to 4, and from 4 to 5 or more.

We selected the series for Slovakia as a case study to analyze. Of all the countries currently in the data base, Slovakia has one of the longest time series of data (1950–2014)[3] and one of the most dramatic declines in total fertility rates (TFR), from TFR = 3.6 in 1950 to TFR = 1.5 in 2014. Over the same period, life exectancy at birth increased from 62.5 to 80.3 years. The mean age at birth changed little, from 28.6 to 29.1 years.

Mortality schedules were extracted from the Human Mortality Database (2019); the age-specific period probabilities of death q_x were used to create the age advancement matrices $\mathbf{D}_j$ in equation (39).

We will show results for the age×paritydistribution of kin, the numbers of kin, the proportional parity structure of kin, and the prevalence of parity-0 and of high parity kin. We will present results for the core of the kinship network (daughters, mothers, and sisters), and also for aunts. For the curious reader with a desire for completeness, a set of all results for all kin types is provided in the Online Supplemental Materials.

As was the case for the age-classified analysis of Japan in Caswell (2019), these results should be considered as an example of the results obtained from an age×parity-classified model comparing demographic situations that differ in both fertility and longevity, not as a detailed analysis of the demography of Slovakia. For convenience, we will describe the results as applying to Slovakia in 1960 or 2014 or some other year, rather than the more cumbersome description of a stable population associated with the demographic rates of Slovakia in 1960, 2014, or whatever.

[3]The HFD cautions that the data before 1960 are not reliable, so we chose to examine only 1960–2014.

5.2.1 Parity distribution of Focal

Since Focal is assumed to be alive at each age, only her proportional parity distribution is relevant. Figure 3 shows how the proportion of high parity individuals declined, and that of low parity individuals increased, between 1960 and 2014. Note that the parity distribution becomes stable after the end of reproduction (age 45 or so). Individuals no longer move among parity classes, and because mortality is not parity-dependent, there is no change in the proportional structure after the end of reproduction.

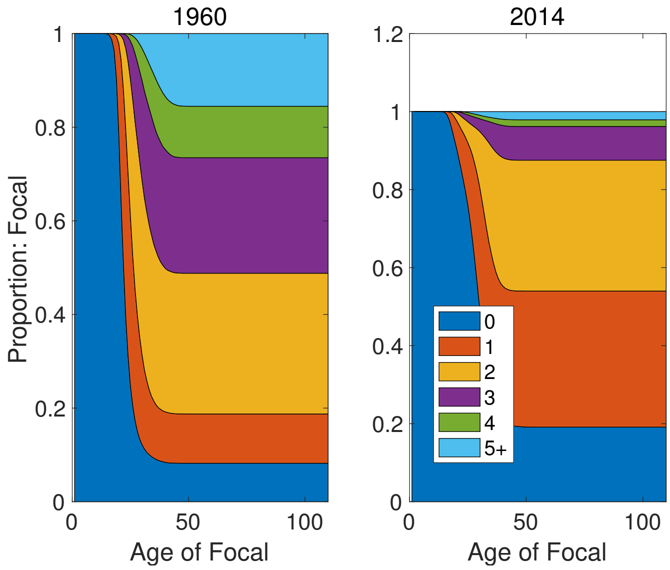

Figure 3: Expected parity distribution of Focal as a function of age, in 1960 and 2014.

5.2.2 Numbers and parity distribution of kin over time

Figures 4 – 7 show the numbers of daughters, mothers, sisters, and aunts over time, classified by parity, at ages 20 and 60 years of Focal. The number of daughters fell over time; at age 20 Focal's daughters have not yet reproduced, at age 60 Focal's daughters cover the complete range of parity, but by 2014, almost all daughters are in parity classes 0, 1, or 2.

The mothers of Focal (since Focal has at most one mother, the expected number of mothers is the probability of having a living mother) survive better in 2014 than in 1960 (no surprise), and there is a clear shift of parity composition, with an increase in parity classes 1and 2. The same is true for the sisters of Focal, which decrease in abundance over time. The aunts of Focal exhibit similar patterns to the sisters of Focal (cf. Figures 7 and 7).

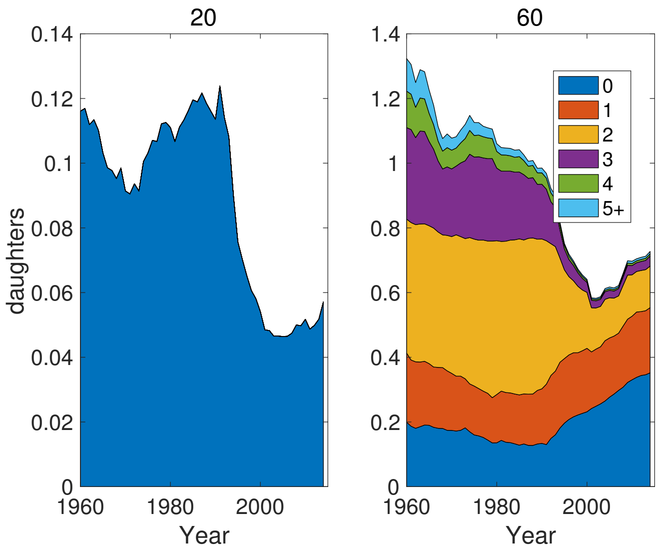

Figure 4: Expected number and parity distribution of daughters as a function of time, for ages 20 and 60 of Focal

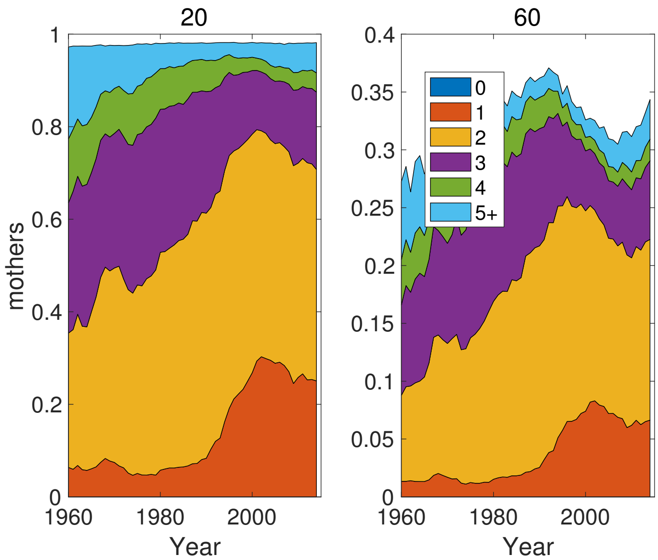

Figure 5: Expected number and parity distribution of mothers as a function of time, for ages 20 and 60 of Focal

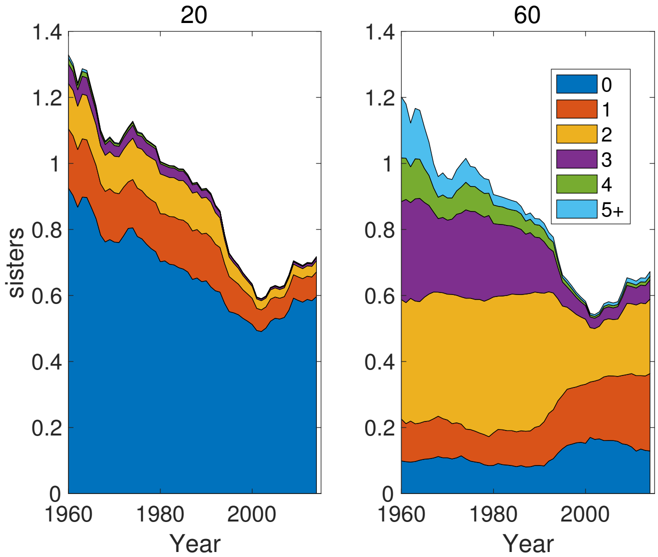

Figure 6: Expected number and parity distribution of sisters as a function of time, for ages 20 and 60 of Focal

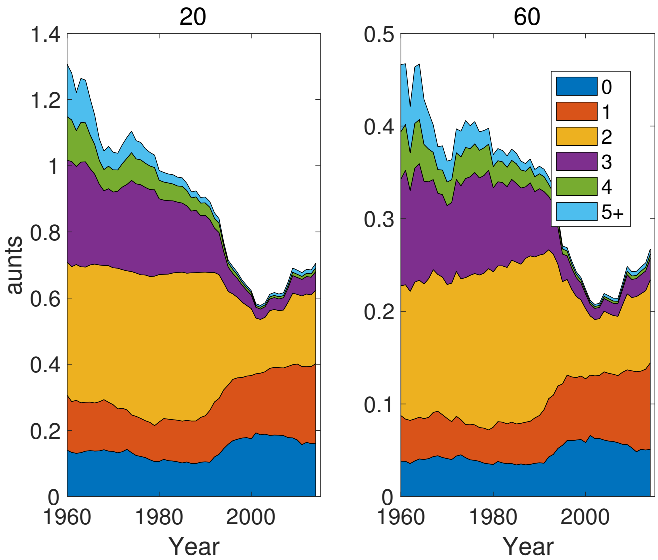

Figure 7: Expected number and parity distribution of aunts as a function of time, for ages 20 and 60 of Focal

5.2.3 Numbers and parity by age of Focal

Now focus on the numbers and parity distribution of kin as a function of the age of Focal, in the years 1960 and 2014. In 2014, Focal has fewer daughtes, and fewer daughters at high parity classes, than in 1960. The survival of mothers is slightly better in 2014, and the reduction in high parity mothers is apparent. The pattern for sisters is similar to that for daughters. In 2014, Focal has onlyabout half the number of aunts as in 1960, and again with fewer high parity individals.

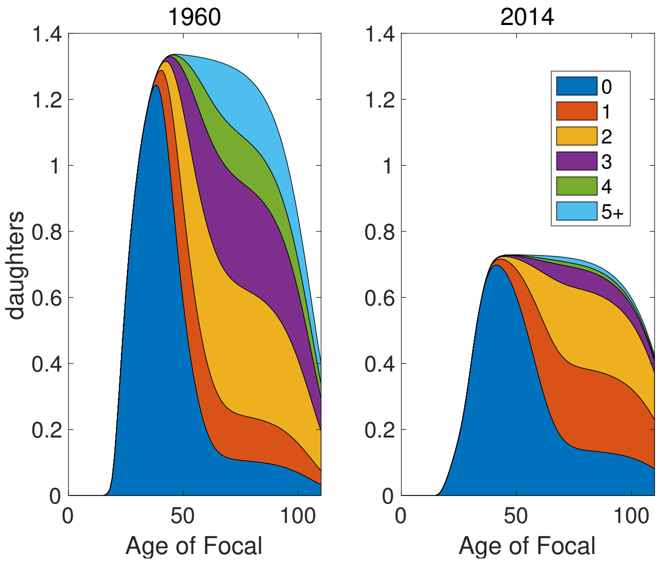

Figure 8: Expected number and parity distribution of daughters as a function of the age of Focal

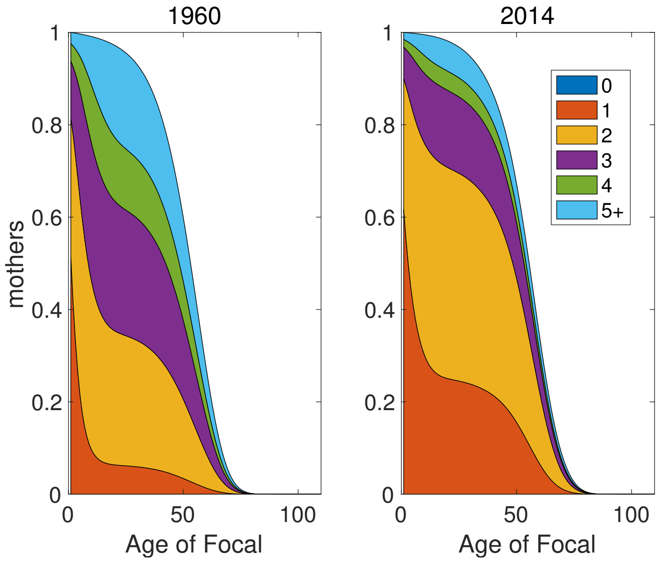

Figure 9: Expected number and parity distribution of mothers as a function of the age of Focal

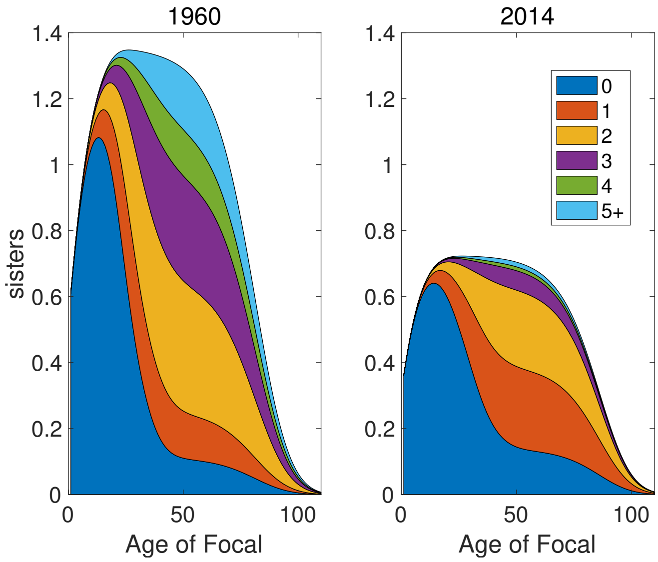

Figure 10: Expected number and parity distribution of sisters as a function of the age of Focal

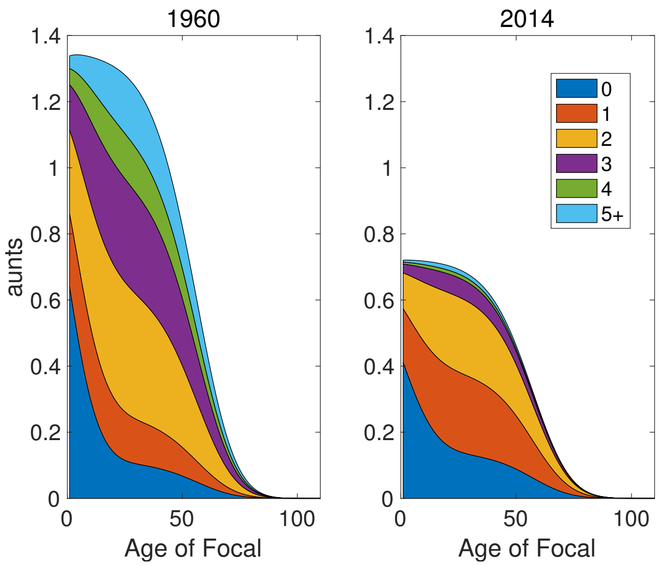

Figure 11: Expected number and parity distribution of aunts as a function of the age of Focal

5.2.4 Proportional parity distributions

From these figures, it is apparent that there have been changes in the proportional parity structure between 1960 and 2014, and over the lifetime of Focal. This section collects plots of the proportional, rather than the absolute, parity distribution for selected kin. These patterns may have important implicaions for the provision of care within families. In all four cases, there has been a dramatic increase in the proportion of low parity, and a decrease in the proportion of high parity kin. For example, in 2014, when Focal is 25 years old, she is more than four times as likely to be an only child (i.e., for her mother to be in parity class 1) than was the case in 1960. Similarly, sisters and aunts in parity classes 0 and 1 are almost three times as likely in 2014, compared to 1960.

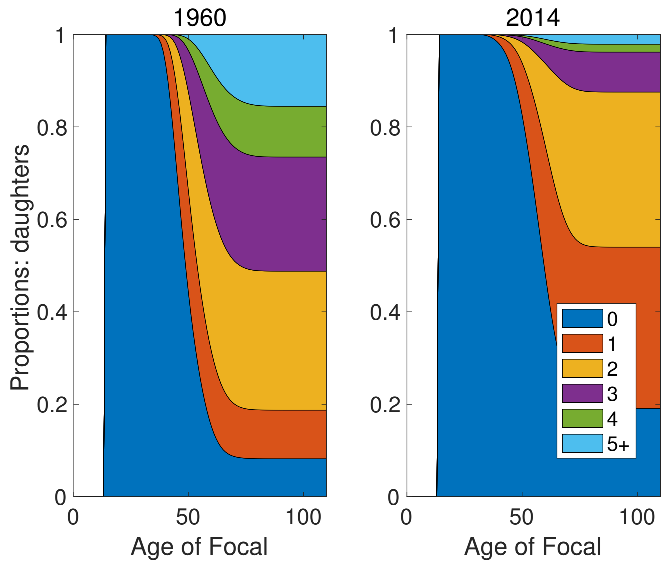

Figure 12: Expected parity distribution, daughters, as function of age of Focal

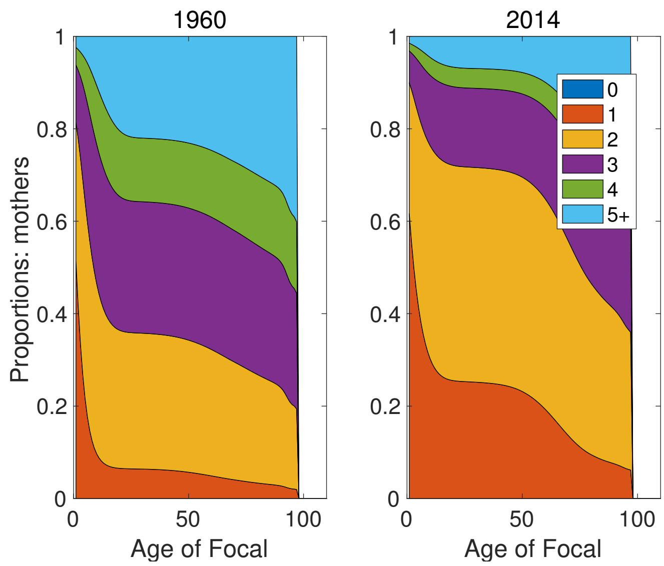

Figure 13: Expected parity distribution, mothers, as function of age of Focal

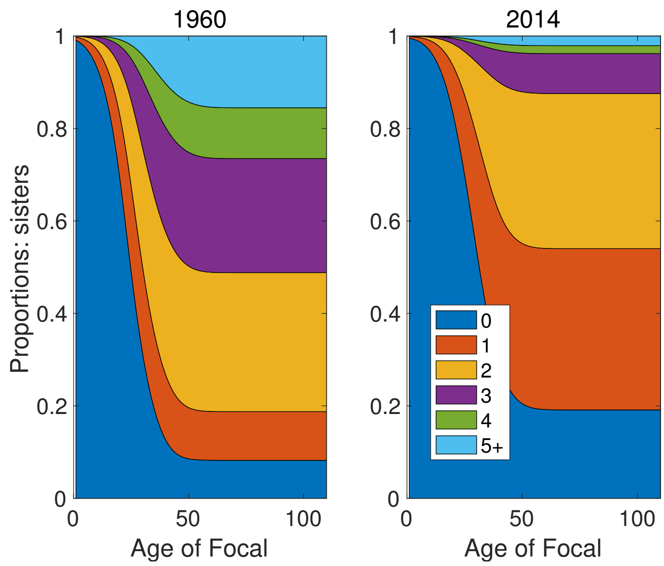

Figure 14: Expected parity distribution, sisters, as function of age of Focal

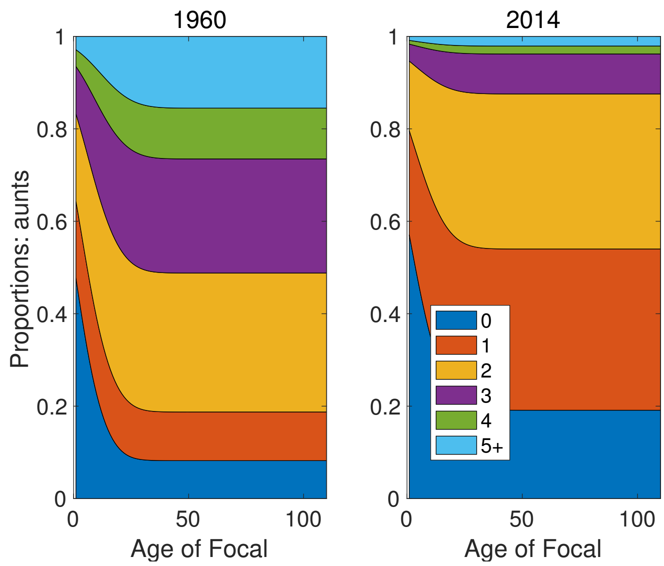

Figure 15: Expected parity distribution, aunts, as function of age of Focal

6 Discussion

7 Acknowledgments

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