Dynamics of the Gini coefficient of the life table

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Abstract

Lifespan variation or lifespan inequality has increasingly received attention as health indicator because it represents the uncertainty about the eventual death an individual experiences. In this paper we take a closer look at the Gini coefficient of the life table (G) and provide additional insights to understand how it relates to improvements in mortality. We focus on how changes over time of the Gini coefficient relate to changes in e_o and a new measure called ϑ that relates to perturbation theory. We provide a mathematical foundation of how the Gini coefficient evolves over time and give analytical formulas to find the threshold age that define premature deaths for this indicator in the sense that mortality improvements below this age decreases lifespan variation and increase e_o. These results provide important implications for understanding trends of lifespan variation over time and age.

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1. Introduction

The life table is an essential tool in mortality studies. It summarizes the current mortality experience of a population and it is usually represented by life expectancy at birth (e_o): the average years a new born individual is expected to survive given the current mortality conditions (Preston et al., 2001). However, life expectancy, as an average, masks variation in lifespans. Lifespan variation or lifespan inequality has increasingly received attention as health indicator because it represents the uncertainty about the eventual death an individual experiences (van Raalte et al., 2018). There exist several indicators to measure lifespan variation, such as the entropy of the life table (Keyfitz, 1977; Demetrius, 1978; Fernández and Beltrán-Sánchez, 2015), the standard deviation or variance of the age-at-death distribution (Tuljapurkar and Edwards, 2011), the coefficient of variation (Aburto et al., 2018), years of life lost (Vaupel et al., 2011), or the Gini coefficient (Shkolnikov et al., 2003).

In this paper we take a closer look at the Gini coefficient of the life table (G) and provide additional insights to understand how it relates to improvements in mortality. We focus on how changes over time of the Gini coefficient relate to changes in e_o and a new measure called ϑ that relates to perturbation theory. We provide a mathematical foundation of how the Gini coefficient evolves over time and give analytical formulas to find the threshold age that define premature deaths for this indicator in the sense that mortality improvements below this age decreases lifespan variation and increase e_o. These results provide important implications for understanding trends of lifespan variation over time and age.

2. The Gini coefficient

The Gini coefficient is one of the most popular indices employed in social science to measure concentration in the distribution of a non-negative random variable (Gini, 1912, 1914). Originally proposed by economists to measure income or wealth inequality, this coefficient has been recently employed in demography and survival analysis to investigate within-group inequality in terms of ages at death (see, for instance, Hanada, 1983; Shkolnikov et al., 2003; Bonetti et al., 2009; Gigliarano et al., 2017).

2.1. Definition

As thoroughly discussed by Yitzhaki and Schechtman (2013), there are several equivalent ways to define the Gini coefficient. Let X be a non-negative random variable with probability density function $f(x)$ and expected value $\mathbb{E}[X]$, one common definition is

$$G = \frac{1}{2\,\mathbb{E}[X]} \int_0^\infty \int_0^\infty |x_1 - x_2|\, f(x_1)\, f(x_2)\, dx_1\, dx_2\ .$$

Accordingly, if X is a random variable of the ages at death in a population, the Gini coefficient expresses the average of absolute differences in individual lifespans relative to the mean length of life $\mathbb{E}[X]$.

Michetti and Dall'Aglio (1957), and later Hanada (1983), suggested a reformulation of the Gini coefficient in terms of the life table functions, given by

$$G = 1 - \frac{\int_0^\infty \ell(x,t)^2\, dx}{\int_0^\infty \ell(x,t)\, dx} = 1 - \frac{\vartheta}{e_o}\ , \qquad (1)$$

where $\ell(x,t)$ is the life table survival function at time t, $e_o = \int_0^\infty \ell(x,t)\, dx$ the life expectancy at birth at time t, and $\vartheta = \int_0^\infty \ell(x,t)^2\, dx$ is the resulting life expectancy of doubling the hazard at all ages. Barthold Jones et al. (2018) interpret ϑ as a measure of *shared life expectancy*, that is, the average time that two newborns at time t are expected to survive together. For the purposes of this article, the definition of the Gini coefficient in (1) will be used in in the following.

2.2. Main properties

The Gini coefficient takes values between 0 and 1, and can be interpreted as a *measure of inequality*. A value of 0 denotes equality in ages at death, i.e. when every individual in the population has the same length of life. The index increases approaching 1 as lifespans become more spread and unequal in the population. This makes the interpretation easy and intuitive: higher values correspond to greater within-group inequality in ages at death.

An additional attractive feature of the Gini coefficient is that it fulfills three important properties for inequality indices (Sen, 1973; Anand, 1983): (i) it does not change if the number of individuals at each age at death is changed by the same proportion (*population-size independence*); (ii) it does not change if each individual lifespan is changed by the same proportion (*scale independence*): (iii) it decreases if years of life are transferred from a longer to a shorter lived individual (*Pigou-Dalton condition*). Note that property (i) allows straightforward comparison between populations, including comparisons between different species (Wrycza et al., 2015). Furthermore, the coefficient is not too sensitive to redistributions at early ages of life, and it reflects well changes at adult ages (Shkolnikov et al., 2003). As such, several authors have chosen the Gini coefficient over other measures to study lifespan inequality, such as... ??

Finally, by being bounded between 0 and 1, the Gini coefficient easily allows switching from a *measure of inequality* to a *measure of equality* of lifespans. In particular, from (1) it is immediate to derive "Drewnowski's index", as coined by Hanada (1983) and defined as

$$\mathscr{D} = 1 - G = \frac{\vartheta}{e_o} = \frac{\int_0^\infty \ell(x,t)^2 \, dx}{\int_0^\infty \ell(x,t) \, dx} \; . \tag{2}$$

This index can be interpreted as a measure of lifespan equality, and shares the same important properties of G. According to Hanada (1983), it was first proposed on a working group on health indicators of the World Health Organization in the early 1980s.

3. Changes over time in Drewnowski's index

In order to analyze changes over time in the Gini coefficient or its equivalent Drewnowski's index, we aim to find an analytical expression for the time derivative of $\mathscr{D}$. In the following, a dot over a function will denote the partial derivative with respect to time, although variable t will be omitted for simplicity.

3.1. Relative derivative of $\mathscr{D}$

Proposition 1. *Let $\mathscr{D} = \vartheta \, / \, e_o$ be Drewnowski's index, where $\vartheta = \int_0^\infty \ell(x)^2 \, dx$, $e_o = \int_0^\infty \ell(x) \, dx$ is the life expectancy at birth, and $\ell(x)$ the probability of surviving from birth to age x. Then, relative changes over time in $\mathscr{D}$ are given by*

$$\frac{\dot{\mathscr{D}}}{\mathscr{D}} = \frac{\dot{\vartheta}}{\vartheta} - \frac{\dot{e}_o}{e_o} \; . \tag{3}$$

Proof. Note that $\mathscr{D} = \vartheta \,/\, e_o$ implies that $\mathscr{D}\, e_o - \vartheta = 0$. Differentiating with respect to time yields

$$\dot{\mathscr{D}}\, e_o + \mathscr{D}\, \dot{e}_o - \dot{\vartheta} = 0 \; .$$

Solving for $\dot{\mathscr{D}}$ and dividing both sides by $\mathscr{D}$, we get (3). □

Equation (3) decomposes relative changes in $\mathscr{D}$ into relative changes of the shared life expectancy between two individuals ϑ, and relative changes in the life expectancy at birth e_o.

3.2. Time derivatives of e_o and ϑ

Vaupel and Canudas-Romo (2003) showed that changes over time in life expectancy at birth are a weighted average of the total rates of mortality improvements, expressed as

$$\dot{e}_o = \int_0^\infty \rho(x)\, w(x)\, dx \; . \tag{4}$$

Function $\rho(x) = -\dot{\mu}(x) \,/\, \mu(x)$ stands for the age-specific rates of mortality improvement, where $\mu(x)$ is the force of mortality (hazard rate) at age x. The weights $w(x) = \mu(x)\, \ell(x)\, e(x)$ are a measure of the importance of death at age x, where $e(x) = \int_x^\infty \ell(a)\, da \,/\, \ell(x)$ is the remaining life expectancy at age x. Following a similar approach, we aim to express the time derivative of ϑ as a weighted average of mortality improvements, but with different weights.

Proposition 2. *Let $\vartheta = \int_0^\infty \ell(x)^2\, dx$, where $\ell(x)$ is the probability of surviving from birth to age x. Then, its partial derivative with respect to time can be expressed as*

$$\dot{\vartheta} = \int_0^\infty \rho(x)\, w(x)\, 2\, \mathscr{D}(x)\, dx \; , \tag{5}$$

where $\rho(x)$ are the age-specific rates of mortality improvement, $w(x)$ the same weights defined in (4)*, and*

$$\mathscr{D}(x) = \frac{\int_x^\infty \ell(a)^2 \, da}{\int_x^\infty \ell(a) \, da} \; .$$

Proof. Applying the chain rule, the derivative of ϑ with respect to time is simply

$$\dot{\vartheta} = \int_0^\infty 2 \, \ell(x) \, \dot{\ell}(x) \, dx \; .$$

Using that $\dot{\ell}(x) = -\ell(x) \int_0^x \dot{\mu}(a) \, da$, and reversing the order of integration, we get

$$\begin{aligned} \dot{\vartheta} &= -2 \int_0^\infty \ell(x)^2 \int_0^x \dot{\mu}(a) \, da \, dx = -2 \int_0^\infty \dot{\mu}(a) \int_a^\infty \ell(x)^2 \, dx \, da \\ &= 2 \int_0^\infty \rho(x) \, \mu(x) \, \ell(x) \, e(x) \, \frac{\int_x^\infty \ell(a)^2 \, da}{\int_x^\infty \ell(a) \, da} \, dx \\ &= \int_0^\infty \rho(x) \, w(x) \, 2 \, \mathscr{D}(x) \, dx \; , \end{aligned}$$

where $w(x) = \mu(x) \, \ell(x) \, e(x)$, which proves (5). □

3.3. Changes over time in $\mathscr{D}$ in terms of mortality improvements

Equations (4) and (5) allow expressing changes over time in $\mathscr{D}$ in terms of mortality improvements. Replacing (4) and (5) in (3) yields

$$\begin{aligned} \dot{\mathscr{D}} &= \mathscr{D} \left(\frac{\dot{\vartheta}}{\vartheta} - \frac{\dot{e}_o}{e_o} \right) \\ &= \mathscr{D} \int_0^\infty \rho(x) \, w(x) \left[\frac{2 \, \mathscr{D}(x)}{\vartheta} - \frac{1}{e_o} \right] dx \\ &= \int_0^\infty \rho(x) \, w(x) \, \frac{2 \, \mathscr{D}(x) - \mathscr{D}}{e_o} \, dx \\ &= \int_0^\infty \rho(x) \, w(x) \, W(x) \, dx \; . \end{aligned} \tag{6}$$

This result shows that changes over time in $\mathscr{D}$ (and analogously in G) are a total average of mortality improvements weighted by $w(x)\,W(x)$, where $w(x) = \mu(x)\,\ell(x)\,e(x)$ are the same weights as in (4) and

$$W(x) = \frac{2\,\mathscr{D}(x) - \mathscr{D}}{e_o}\,.$$

4. The threshold age

4.1. Positive and negative contributions to lifespan equality

Because Drewnowski's index is a measure of equality, $\dot{\mathscr{D}} > 0$ indicates that lifespan equality increases over time, whereas $\dot{\mathscr{D}} < 0$ implies that lifespan equality decreases over time, amplifying the variation of ages at death. Equation (6) can then be used to analyze the existence of a threshold age that separates *positive* form *negative* contributions to lifespan equality as a result of mortality improvements.

Note that in the assumption that mortality improvements occur at all ages, $\rho(x) = -\dot{\mu}(x)\,/\,\mu(x) > 0$ is a strictly positive function. Therefore, from (6),

1. Those ages x for which $w(x)\,W(x) > 0$ will contribute *positively* to Drewnowski's index $\mathscr{D}$ and increase lifespan equality;
2. Those ages x for which $w(x)\,W(x) < 0$ will contribute *negatively* to Drewnowski's index $\mathscr{D}$ and favor lifespan inequality;
3. Those ages x for which $w(x)\,W(x) = 0$ will have no effect on the variation over time of $\mathscr{D}$.

Any existing threshold age that separates positive from negative contributions to lifespan equality will occur whenever $w(x)\,W(x) = 0$. Since $\mu(x)$,

$\ell(x)$, and $e(x)$ are all positive functions, so are $w(x)$ and e_o. Hence,

$$w(x)\, W(x) = 0 \quad \iff \quad 2\,\mathscr{D}(x) - \mathscr{D} = 0\,. \tag{7}$$

4.2. Existence and uniqueness of the threshold age

By means of the following two propositions and one theorem, we aim to prove that in a scenario in which mortality improvements occur at all ages and $\rho(x) > 0$ for all $x \geq 0$, there exists a unique threshold age $a^{\mathcal{D}}$ that separates positive from negative contributions to lifespan equality (measured by $\mathscr{D}$) as a result of those improvements.

Remark. *Following* (2)*, Drewnowski's index* $\mathscr{D}$ *is bounded between 0 and 1, reaching a value of 1 when there is complete equality in the ages at death within a population. A score of 0 would express that there is complete inequality in the ages at death, but by definition this value can never be reached:*

$$\mathscr{D} = 0 \iff \frac{\int_0^\infty \ell(x)^2\, dx}{\int_0^\infty \ell(x)\, dx} = 0 \iff \int_0^\infty \ell(x)^2\, dx = 0 \iff \ell(x) = 0 \tag{8}$$

for all ages $x \geq 0$*. But this implies that the denominator in* (8) *is also 0 because* $\ell(x) \geq 0$ *is always positive, and therefore* $\mathscr{D}$ *would be undefined. Hence,* $0 < \mathscr{D} \leq 1$*.*

Proposition 3. *Let* $\ell(x)$ *be the probability of surviving from birth to age* x*,* $\mathscr{D}$ *Drewnowski's index as defined in* (2)*, and* $\mathscr{D}(x) = \int_x^\infty \ell(a)^2\, da \,/ \int_x^\infty \ell(a)\, da$*. Define the function* $g(x) := 2\,\mathscr{D}(x) - \mathscr{D}$*. Then, there exists at least one age* $a^{\mathcal{D}}$ *such that* $g(a^{\mathcal{D}}) = 0$*.*

Proof. At age $x = 0$,

$$g(0) = 2\,\mathscr{D}(0) - \mathscr{D} = 2\,\mathscr{D} - \mathscr{D} = \mathscr{D} > 0 \tag{9}$$

by definition, since $0 < \mathscr{D} \leq 1$.

Besides, when ages become arbitrarily large,

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} (2\,\mathscr{D}(x) - \mathscr{D}) = 2 \lim_{x\to\infty} \mathscr{D}(x) - \mathscr{D} \ ,$$

which only depends on the behavior of $\mathscr{D}(x)$. Because $\ell(x) \in [0,1]$ for all ages $x \geq 0$, we have that $0 \leq \ell(x)^2 \leq \ell(x)$ and

$$0 \leq \lim_{x\to\infty} \int_x^{\infty} \ell(a)^2\, da \leq \lim_{x\to\infty} \int_x^{\infty} \ell(a)\, da = \lim_{x\to\infty} e(x)\, \ell(x) = 0 \ ,$$

where $e(x)$ is the remaining life expectancy at age x, which proves that both integrals tend to 0 as x approaches ∞. Consequently, the following limit

$$\lim_{x\to\infty} \mathscr{D}(x) = \lim_{x\to\infty} \frac{\int_x^{\infty} \ell(a)^2\, da}{\int_x^{\infty} \ell(a)\, da}$$

is indeterminate, but applying L'Hôpital's rule, we get

$$\lim_{x\to\infty} \mathscr{D}(x) = \lim_{x\to\infty} \frac{\frac{\partial}{\partial x} \int_x^{\infty} \ell(a)^2\, da}{\frac{\partial}{\partial x} \int_x^{\infty} \ell(a)\, da} = \lim_{x\to\infty} \frac{-\ell(x)^2}{-\ell(x)} = \lim_{x\to\infty} \ell(x) = 0 \ .$$

As a result,

$$\lim_{x\to\infty} g(x) = 2 \lim_{x\to\infty} \mathscr{D}(x) - \mathscr{D} = -\mathscr{D} < 0 \ . \tag{10}$$

Finally, using (9) and (10), in a continuous framework the intermediate value theorem guarantees the existence of at least one positive age $a^{\mathcal{D}}$ at which $g(a^{\mathcal{D}}) = 0$. □

Proposition 4. *Let $\ell(x)$ be the probability of surviving from birth to age x, $\mathscr{D}$ Drewnowski's index as defined in* (2)*, and $\mathscr{D}(x) = \int_x^{\infty} \ell(a)^2\, da \,/\, \int_x^{\infty} \ell(a)\, da$. Then, $g(x) := 2\,\mathscr{D}(x) - \mathscr{D}$ is a strictly decreasing function.*

Proof. In order to demonstrate that $g(x)$ is a strictly decreasing function, it suffices to show that its first derivative is negative for all ages $x \geq 0$. Note that since $\mathscr{D}$ does not depend on age,

$$\frac{\partial}{\partial x}\, g(x) < 0 \quad \Longleftrightarrow \quad \frac{\partial}{\partial x}\, \mathscr{D}(x) < 0\,.$$

Applying the quotient rule together with the fundamental theorem of calculus, we get

$$\begin{aligned}
\frac{\partial}{\partial x}\, \mathscr{D}(x) &= \frac{\partial}{\partial x} \left(\frac{\int_x^\infty \ell(a)^2\, da}{\int_x^\infty \ell(a)\, da} \right) \\
&= \frac{\int_x^\infty \ell(a)\, da\, \frac{\partial}{\partial x} \left(\int_x^\infty \ell(a)^2\, da\right) - \int_x^\infty \ell(a)^2\, da\, \frac{\partial}{\partial x} \left(\int_x^\infty \ell(a)\, da\right)}{\left(\int_x^\infty \ell(a)\, da\right)^2} \\
&= \frac{\int_x^\infty \ell(a)\, da\, \left(-\ell(x)^2\right) - \int_x^\infty \ell(a)^2\, da\, \left(-\ell(x)\right)}{\left(\int_x^\infty \ell(a)\, da\right)^2}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial x}\, g(x) < 0 &\Longleftrightarrow \ell(x) \int_x^\infty \ell(a)^2\, da - \ell(x)^2 \int_x^\infty \ell(a)\, da < 0 \\
&\Longleftrightarrow \frac{1}{\ell(x)^2} \int_x^\infty \ell(a)^2\, da < \frac{1}{\ell(x)} \int_x^\infty \ell(a)\, da\,.
\end{aligned}$$

Note that $\ell(x) = \exp\left[-\int_0^x \mu(a)\, da\right]$ for a given age-specific hazard function $\mu(x)$. As such, $\ell(x)^2 = \exp\left[-\int_0^x 2\,\mu(a)\, da\right]$ can be interpreted as the survival schedule with doubling hazard $2\,\mu(x)$ at all ages $x \geq 0$. We can then define

$$\tilde{e}(x) = \frac{1}{\ell(x)^2} \int_x^\infty \ell(a)^2\, da$$

as the remaining life expectancy at age x of a population with survival schedule $\ell(x)^2$ and age-specific force of mortality $2\,\mu(x)$. Then,

$$\frac{\partial}{\partial x}\, g(x) < 0 \Longleftrightarrow \frac{1}{\ell(x)^2} \int_x^\infty \ell(a)^2\, da < \frac{1}{\ell(x)} \int_x^\infty \ell(a)\, da \Longleftrightarrow \tilde{e}(x) < e(x)$$

for all $x \geq 0$, which holds true since doubling the hazard corresponds to a lower remaining life expectancy, in the reasonable assumption that $\mu(x) > 0$ at all ages. $\square$

Theorem. *Let $\mathscr{D} = \vartheta \,/\, e_o$ be Drewnowski's index, where $\vartheta = \int_0^\infty \ell(x)^2\, dx$, $e_o = \int_0^\infty \ell(x)\, dx$ is the life expectancy at birth, and $\ell(x)$ the probability of surviving from birth to age x. Assume mortality improvements over time occur at all ages. Then, there exists a unique threshold age $a^{\mathcal{D}}$ that separates positive from negative contributions to lifespan equality, measured by $\mathscr{D}$, as a result of those improvements.*

Proof. Following (6), changes over time in $\mathscr{D}$ can be expressed as a weighted average of mortality improvements, given by

$$\dot{\mathscr{D}} = \int_0^\infty \rho(x)\, w(x)\, W(x)\, dx \;,$$

where $\rho(x)$ are the age-specific rates of mortality improvement over time, and $w(x)\, W(x)$ the weights. By assumption, $\rho(x) > 0$ for all ages $x \geq 0$. Therefore, any threshold age that separates positive from negative contributions to lifespan equality as a result of mortality improvements will occur whenever $w(x)\, W(x) = 0$. From (7),

$$w(x)\, W(x) = 0 \quad \Longleftrightarrow \quad 2\, \mathscr{D}(x) - \mathscr{D} = 0 \;,$$

where $\mathscr{D}(x) = \int_x^\infty \ell(a)^2\, da \,/\, \int_x^\infty \ell(a)\, da$. Proposition 3 proves the existence of at least one positive age $a^{\mathcal{D}}$ at which $2\, \mathscr{D}(a^{\mathcal{D}}) - \mathscr{D} = 0$. In addition, from Proposition 4 the function $g(x) := 2\, \mathscr{D}(x) - \mathscr{D}$ is strictly decreasing. Hence, assuming continuity, $g(x) := 2\, \mathscr{D}(x) - \mathscr{D}$ is a one-to-one function, and therefore the threshold age $a^{\mathcal{D}}$ is unique. $\square$

5. Application

The following steps consist on applying our framework to the best practice lifespan variation.

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